

On progressive filtration expansion with a process

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Abstract

In this paper we study progressive filtration expansions with càdlàg processes. Using results from the weak convergence of σ -fields theory, we first establish a semimartingale convergence theorem. Then we apply it in a filtration expansion with a process setting and provide sufficient conditions for a semimartingale of the base filtration to remain a semimartingale in the expanded filtration. Finally, an application to the expansion of a Brownian filtration with a time reversed diffusion is given through a detailed study.

1 Introduction

One of the key insights of K. Itô when he developed the Itô integral was to restrict the space of integrands to what we now call predictable processes. This allowed the integral to have a type of bounded convergence theorem that N. Wiener was unable to obtain with unrestricted random integrands. The Itô integral has since been extended to general semimartingales. If one tries however to expand (i.e. to enlarge) the filtration, then one is playing with fire, and one may lose the key properties Itô originally obtained with his restriction to predictable processes. In the 1980's a theory of such filtration expansions was nevertheless successfully developed for two types of expansion: initial expansions and progressive expansions; see for instance [10] and [12], or the more recent partial exposition in [15, Chapter VI]. The initial expansion of a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ with a random variable τ is the filtration \mathbb{H} obtained as the right-continuous modification of $(\mathcal{F}_t \vee \sigma(\tau))_{t \geq 0}$. The progressive expansion \mathbb{G} is obtained as any right-continuous filtration containing \mathbb{F} and making τ a stopping time. When referring to the progressive expansion with a random variable in this paper, we mean the smallest such filtration. One is usually interested in the cases where \mathbb{F} semimartingales remain semimartingales in the expanded filtrations and in their decompositions when viewed as semimartingales in the expanded filtrations. The subject has regained interest recently, due to applications in Mathematical Finance, as exemplified (for example) in the work of Jeanblanc and Le Cam [11].

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In this article we go beyond the simple cases of initial expansion and progressive expansion with a random variable. Instead we consider the (more complicated) case of expansion of a filtration through dynamic enlargement, by adding a stochastic process as it evolves simultaneously to the evolution of the original process. In order to do this, we begin with simple cases where we add marked point processes, and then we use the theory of the convergence of sigma fields recently developed by Antonelli, Coquet, Kohatsu-Higa, Mackevicius, Mémin, and Slominski (see [1],[5],[6]) to obtain more sophisticated enlargement possibilities. We combine the convergence results with an extension of an old result of Barlow and Protter [3], finally obtaining the key results, which include the forms of the semimartingale decompositions in the enlarged filtrations. We then apply these results to an example where we enlarge the filtration with another process which is evolving backwards in time. To do this we need to use density estimates inspired by the work of Bally and Talay [2].

The techniques developed in this paper require a long preliminary treatment of the convergence of σ fields, and to a lesser extent the convergence of filtrations. This delays the key theorems such that they occur rather late in the paper, so perhaps it is wise to indicate that the main results of interest (in the authors' opinion) are Theorems 6 and 8, which show how one can expand filtrations with processes and have semimartingales remain semimartingales in the enlarged filtrations. The authors also wish to mention here that the example provided in Theorem 10 shows how the hypotheses (perhaps a bit strange at first glance) of Theorem 8 can arise naturally in applications, and it shows the potential utility of the results of this paper. That said, the preliminary results on the weak convergence of σ fields has an interest in their own right.

1.1 Previous Results

For the initial expansion \mathbb{H} of a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ with a random variable τ , one well-known situation is when *Jacod's criterion* is satisfied (see [10] or alternatively [15, Theorem 10, p. 371]), and as far as one is concerned by the progressive expansion, filtration \mathbb{G} , this always holds up to the random time τ as proved by Jeulin and Yor and holds on all $[0, \infty)$ for honest times (see [12]). In both [13] and [11], this is proved to hold also for random times satisfying *Jacod's criterion*. In [13], the authors link the two previous types of expansions and are able to provide similar results for more general types of expansion of filtrations. They extend for instance these results to the multiple time case, without any restrictions on the ordering of the individual times and more importantly to the filtration expanded by a counting process $N_t^n = \sum_{i=1}^n X_i 1_{\{\tau_i \leq t\}}$, i.e. the smallest right-continuous filtration containing \mathbb{F} and to which the process N^n is adapted.

For a given filtration \mathbb{F} and a given càdlàg process X , the smallest right-continuous filtration containing \mathbb{F} and to which X is adapted will be called the progressive expansion of \mathbb{F} with X . In this paper we pursue the analysis started in [13] and investigate the stability of the semimartingale property of \mathbb{F} semimartingales in progressive expansions of \mathbb{F} with càdlàg processes X . We apply the results in [13] together with results from the theory of weak convergence of σ -fields (see [5] and [6]) to obtain a general criterion that

guarantees this property, at least for \mathbb{F} semimartingales satisfying suitable integrability assumptions. Hoover [9], following remarks by M. Barlow and S. Jacka, introduced the weak convergence of σ -fields and of filtrations in 1991. The next big step was in 2000 with the seminal paper of Antonelli and Kohatsu-Higa[1]. This was quickly followed by the work of Coquet, Mémmin and Mackevicius [6] and by Coquet, Mémmin and Slominsky [5]. We will recall fundamental results on the topic but we refer the interested reader to [5] and [6] for details. In these papers, all filtrations are indexed by a compact time interval $[0, T]$. We work within the same framework and assume that a probability space (Ω, \mathcal{H}, P) and a positive integer T are given. All filtrations considered in this paper are assumed to be completed by the P -null sets of \mathcal{H} . By the natural filtration of a process X , we mean the right-continuous filtration associated to the natural filtration of X . The concepts of weak convergence of σ -fields and of filtrations rely on the topology imposed on the space of càdlàg processes and we use the Skorohod J_1 topology as it is done in [5].

1.2 Outline

An outline of this paper is the following. In section 2, we recall basic facts on the weak convergence of σ -fields and establish fundamental lemmas for subsequent use. The last subsection provides a sufficient condition for the semimartingale property to hold for a given càdlàg adapted process based on the weak convergence of σ -fields. The sufficient condition we provide at this point is unlikely to hold in a filtration expansion context, however the proof of this result underlines what can go wrong under the more natural assumptions considered in the next section.

Section 3 extends the main theorem in [3] and proves a general result on the convergence of \mathbb{G}^n special semimartingales to a \mathbb{G} adapted process X , where $(\mathbb{G}^n)_{n \geq 1}$ and \mathbb{G} are filtrations such that \mathcal{G}_t^n converges weakly to \mathcal{G}_t for each $t \geq 0$. The process X is proved to be a \mathbb{G} special semimartingale under sufficient conditions on the regularity of the local martingale and finite variation parts of the \mathbb{G}^n semimartingales. This is then applied to the case where the filtrations \mathbb{G}^n are obtained by progressively expanding a base filtration \mathbb{F} with processes N^n converging in probability to some process N . We provide sufficient conditions for an \mathbb{F} semimartingale to remain a \mathbb{G} semimartingale, where \mathbb{G} is the progressive expansion of \mathbb{F} with N .

Section 4 applies the results obtained in Section 3 to the case where the base filtration \mathbb{F} is progressively expanded by a càdlàg process whose increments satisfy a *generalized Jacod's criterion* with respect to the filtration \mathbb{F} along *some* sequence of subdivisions whose mesh tends to zero. An application to the expansion of a Brownian filtration with a time reversed diffusion is given through a detailed study, and the canonical decomposition of the Brownian motion in the expanded filtration is provided. Finally, a possible application to stochastic volatility models is suggested.

2 Weak convergence of σ -fields and filtrations

2.1 Definitions and fundamental results

Let \mathbb{D} be the space of càdlàg¹ functions from $[0, T]$ into \mathbb{R} . Let Λ be the set of time changes from $[0, T]$ into $[0, T]$, i.e. the set of all continuous strictly increasing functions $\lambda : [0, T] \rightarrow [0, T]$ such that $\lambda(0) = 0$ and $\lambda(T) = T$. We define the Skorohod distance as follows

$$d_S(x, y) = \inf_{\lambda \in \Lambda} \{ \|\lambda - Id\|_\infty \vee \|x - y \circ \lambda\|_\infty \}$$

for each x and y in \mathbb{D} . Let $(X^n)_{n \geq 1}$ and X be càdlàg processes (i.e. whose paths are in \mathbb{D}), indexed by $[0, T]$ and defined on (Ω, \mathcal{H}, P) . We will write $X^n \xrightarrow{P} X$ when $(X^n)_{n \geq 1}$ converges in probability under the Skorohod J_1 topology to X i.e. when the sequence of random variables $(d_S(X^n, X))_{n \geq 1}$ converges in probability to zero. We can now introduce the concepts of weak convergence of σ -fields and of filtrations.

Definition 1 *A sequence of σ -algebras \mathcal{A}^n converges weakly to a σ -algebra \mathcal{A} if and only if for all $B \in \mathcal{A}$, $E(1_B | \mathcal{A}^n)$ converges in probability to 1_B . We write $\mathcal{A}^n \xrightarrow{w} \mathcal{A}$.*

Definition 2 *A sequence of right-continuous filtrations \mathbb{F}^n converges weakly to a filtration \mathbb{F} if and only if for all $B \in \mathcal{F}_T$, the sequence of càdlàg martingales $E(1_B | \mathcal{F}_t^n)$ converges in probability under the Skorohod J_1 topology on \mathbb{D} to the martingale $E(1_B | \mathcal{F}_t)$. We write $\mathbb{F}^n \xrightarrow{w} \mathbb{F}$.*

The following lemmas provide characterizations of the weak convergence of σ -fields and filtrations. We refer to [5] for the proofs.

Lemma 1 *A sequence of σ -algebras \mathcal{A}^n converges weakly to a σ -algebra \mathcal{A} if and only if $E(Z | \mathcal{A}^n)$ converges in probability to Z for any integrable and \mathcal{A} measurable random variable Z .*

Lemma 2 *A sequence of filtrations \mathbb{F}^n converges weakly to a filtration \mathbb{F} if and only if $E(Z | \mathcal{F}_t^n)$ converges in probability under the Skorohod J_1 topology to $E(Z | \mathcal{F}_t)$, for any integrable, \mathcal{F}_T measurable random variable Z .*

The weak convergence of the σ -fields \mathcal{F}_t^n to \mathcal{F}_t for all t does not imply the weak convergence of the filtrations \mathbb{F}^n to \mathbb{F} . The reverse implication does not hold neither.

Coquet, Mémin and Slominsky provide a characterization of weak convergence of filtrations when the limiting filtration is the natural filtration of some càdlàg process X , see Lemma 3 in [5]. We provide a similar result for weak convergence of σ -fields when the limiting σ -field is generated by some càdlàg process X .

Lemma 3 *Let X be a càdlàg process. Define $\mathcal{A} = \sigma(X_t, 0 \leq t \leq T)$ and let $(\mathcal{A}^n)_{n \geq 1}$ be a sequence of σ -fields. Then $\mathcal{A}^n \xrightarrow{w} \mathcal{A}$ if and only if*

$$E(f(X_{t_1}, \dots, X_{t_k}) | \mathcal{A}^n) \xrightarrow{P} f(X_{t_1}, \dots, X_{t_k})$$

¹French acronym for right-continuous with left limits

for all $k \in \mathbb{N}$, t_1, \dots, t_k points of a dense subset \mathcal{D} of $[0, T]$ containing T and for any continuous and bounded function $f : \mathbb{R}^k \rightarrow \mathbb{R}$.

Proof. Necessity follows from the definition of the weak convergence of σ -fields. Let us prove the sufficiency. Let $A \in \mathcal{A}$ and $\epsilon > 0$. There exists $k \in \mathbb{N}$ and t_1, \dots, t_k in \mathcal{D} such that

$$E(|f(X_{t_1}, \dots, X_{t_k}) - 1_A|) < \epsilon.$$

Let $\eta > 0$. We need to show that $P(|E(1_A | \mathcal{A}^n) - 1_A| \geq \eta)$ converges to zero.

$$\begin{aligned} P(|E(1_A | \mathcal{A}^n) - 1_A| \geq \eta) &\leq P(|E(1_A | \mathcal{A}^n) - E(f(X_{t_1}, \dots, X_{t_k}) | \mathcal{A}^n)| \geq \frac{\eta}{3}) \\ &\quad + P(|E(f(X_{t_1}, \dots, X_{t_k}) | \mathcal{A}^n) - f(X_{t_1}, \dots, X_{t_k})| \geq \frac{\eta}{3}) + P(|f(X_{t_1}, \dots, X_{t_k}) - 1_A| \geq \frac{\eta}{3}) \\ &\leq \frac{6}{\eta} E(|f(X_{t_1}, \dots, X_{t_k}) - 1_A|) + P(|E(f(X_{t_1}, \dots, X_{t_k}) | \mathcal{A}^n) - f(X_{t_1}, \dots, X_{t_k})| \geq \frac{\eta}{3}) \\ &\leq \frac{6}{\eta} \epsilon + P(|E(f(X_{t_1}, \dots, X_{t_k}) | \mathcal{A}^n) - f(X_{t_1}, \dots, X_{t_k})| \geq \frac{\eta}{3}) \end{aligned}$$

where the second inequality follows from the Markov inequality. By assumption, there exists N such that for all $n \geq N$,

$$P(|E(f(X_{t_1}, \dots, X_{t_k}) | \mathcal{A}^n) - f(X_{t_1}, \dots, X_{t_k})| \geq \frac{\eta}{3}) \leq \epsilon$$

hence $P(|E(1_A | \mathcal{A}^n) - 1_A| \geq \eta) \leq (\frac{6}{\eta} + 1)\epsilon$. ■

In [5], the authors provide cases where the weak convergence of a sequence of natural filtrations of given càdlàg processes is guaranteed. We provide here a similar result for point wise weak convergence of the associated σ -fields.

Lemma 4 *Let $(X^n)_{n \geq 1}$ be a sequence of càdlàg processes converging in probability to a càdlàg process X . Let \mathbb{F}^n and \mathbb{F} be the natural filtrations of X^n and X respectively. Then $\mathcal{F}_t^n \xrightarrow{w} \mathcal{F}_t$ for all t such that $P(\Delta X_t \neq 0) = 0$.*

Proof. Let t be such that $P(\Delta X_t \neq 0) = 0$. Since X is càdlàg, there exists $k \in \mathbb{N}$, and $t_1, \dots, t_k \leq t$ such that $P(\Delta X_{t_i} \neq 0) = 0$, for all $1 \leq i \leq k$. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}$ be a continuous and bounded function. By Lemma 3, it suffices to show that

$$E(f(X_{t_1}, \dots, X_{t_k}) | \mathcal{F}_t^n) \xrightarrow{P} f(X_{t_1}, \dots, X_{t_k})$$

An application of Markov's inequality leads to the following estimate

$$\begin{aligned} P(|E(f(X_{t_1}, \dots, X_{t_k}) | \mathcal{F}_t^n) - f(X_{t_1}, \dots, X_{t_k})| \geq \eta) &\leq P(|E(f(X_{t_1}, \dots, X_{t_k}) - f(X_{t_1}^n, \dots, X_{t_k}^n) | \mathcal{F}_t^n)| \geq \frac{\eta}{2}) \\ &\quad + P(|E(f(X_{t_1}^n, \dots, X_{t_k}^n) | \mathcal{F}_t^n) - f(X_{t_1}, \dots, X_{t_k})| \geq \frac{\eta}{2}) \\ &\leq \frac{4}{\eta} E(|f(X_{t_1}^n, \dots, X_{t_k}^n) - f(X_{t_1}, \dots, X_{t_k})|) \end{aligned}$$

Since $X^n \xrightarrow{P} X$ and $P(\Delta X_{t_i} \neq 0) = 0$, for all $1 \leq i \leq k$, it follows that

$$(X_{t_1}^n, \dots, X_{t_k}^n) \xrightarrow{P} (X_{t_1}, \dots, X_{t_k}) \quad (1)$$

and hence $f(X_{t_1}^n, \dots, X_{t_k}^n)$ converges in L^1 to $f(X_{t_1}, \dots, X_{t_k})$. This ends the proof of the lemma. ■

For a given càdlàg process X , a time t such that $P(\Delta X_t \neq 0) > 0$ will be called a fixed time of discontinuity of X , and we will say that X has no fixed times of discontinuity if $P(\Delta X_t \neq 0) = 0$ for all $0 \leq t \leq T$. Lemma 4 can be improved when the sequence X^n is the discretization of the càdlàg process X along some refining sequence of subdivisions $(\pi_n)_{n \geq 1}$ such that each fixed time of discontinuity of X belongs to $\cup_n \pi_n$.

Lemma 5 *Let X be a càdlàg process. Consider a sequence of subdivisions $(\pi_n = \{t_k^n\}, n \geq 1)$ whose mesh tends to zero and let X_n be the discretized process defined by $X_t^n = X_{t_k^n}$, for all $t_k^n \leq t < t_{k+1}^n$. Let \mathbb{F} and \mathbb{F}^n be the natural filtrations of X and X^n . If each fixed time of discontinuity of X belongs to $\cup_n \pi_n$, then $\mathcal{F}_t^n \xrightarrow{w} \mathcal{F}_t$, for all t .*

Proof. The proof is essentially the same as that of Lemma 4. Now, equation (1) holds because the subdivision contains the discontinuity points of X . ■

We will also need the two following lemmas from the theory of weak convergence of σ -fields. The first result is proved in [6] and the second one in [5].

Lemma 6 *Let $(\mathcal{A}^n)_{n \geq 1}$ and $(\mathcal{B}^n)_{n \geq 1}$ be two sequences of σ -fields that weakly converge to \mathcal{A} and \mathcal{B} , respectively. Then*

$$\mathcal{A}^n \vee \mathcal{B}^n \xrightarrow{w} \mathcal{A} \vee \mathcal{B}$$

Lemma 7 *Let $(\mathcal{A}^n)_{n \geq 1}$ and $(\mathcal{B}^n)_{n \geq 1}$ be two sequences of σ -fields such that $\mathcal{A}^n \subset \mathcal{B}^n$ for all n . Let \mathcal{A} be a σ -field. If $\mathcal{A}^n \xrightarrow{w} \mathcal{A}$ then $\mathcal{B}^n \xrightarrow{w} \mathcal{A}$.*

As pointed out in [5], the results in Lemmas 6 and 7 are not true as far as one is interested in weak convergence of filtrations.

2.2 Approximation of a given stopping time

Let $(\mathbb{G}^n)_{n \geq 1}$ be a sequence of right-continuous filtrations and let \mathbb{G} be a right-continuous filtration such that $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$ for all t . In order to obtain our filtration expansion results, we need a key theorem that guarantees the \mathbb{G} semimartingale property of a limit of \mathbb{G}^n semimartingales as in Theorem 3. The following lemma, which permits to approximate any \mathbb{G} bounded stopping time τ by a sequence of \mathbb{G}^n stopping times, will be of crucial importance in the proof of Theorem 3, Part (ii). We prove this result using successive approximations in the case where τ takes a finite number of values and show how this property is inherited by bounded stopping times. We do not study the general case (unbounded stopping times) since we are working on the finite time interval $[0, T]$.

Lemma 8 *Let $(\mathbb{G}^n)_{n \geq 1}$ be a sequence of right-continuous filtrations and let \mathbb{G} be a right-continuous filtration such that $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$ for all t . Let τ be a bounded \mathbb{G} stopping time.*

Then there exists $\phi : \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing and a bounded sequence $(\tau_n)_{n \geq 1}$ such that the subsequence $(\tau_{\phi(n)})_{n \geq 1}$ converges in probability to τ and each $\tau_{\phi(n)}$ is a $\mathbb{G}^{\phi(n)}$ stopping time.

Proof. Let τ be a \mathbb{G} stopping time bounded by T . Then there exists a sequence τ_n of \mathbb{G} stopping times decreasing a.s. to τ and taking values in $\{\frac{k}{2^n}, k \in \{0, 1, \dots, [2^n T] + 1\}\}$. This is true since the sequence $\tau_n = \frac{[2^n \tau] + 1}{2^n}$ obviously works. Hence τ_n takes a finite number of values. We claim that

Claim. for each n , we can construct a sequence $(\tau_{n,m})_{m \geq 1}$ converging in probability to τ_n , and such that $\tau_{n,m}$ is a \mathbb{G}^m stopping time, for each m .

Assume we can do so and let $\eta > 0$ and $\epsilon > 0$. Then for each n , $\lim_{m \rightarrow \infty} P(|\tau_{n,m} - \tau_n| > \frac{\eta}{2}) = 0$, i.e. for each n there exists M_n such that for all $m \geq M_n$, $P(|\tau_{n,m} - \tau_n| > \frac{\eta}{2}) \leq \frac{\epsilon}{2}$. Define $\phi(1) = M_1$ and $\phi(n) = \max(M_n, \phi(n-1) + 1)$ by induction. The application $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is strictly increasing, and for each n , $\tau_{n,\phi(n)}$ is a $\mathbb{G}^{\phi(n)}$ stopping time and $P(|\tau_{n,\phi(n)} - \tau_n| > \frac{\eta}{2}) \leq \frac{\epsilon}{2}$. It follows that

$$P(|\tau_{n,\phi(n)} - \tau| > \eta) \leq P(|\tau_{n,\phi(n)} - \tau_n| > \frac{\eta}{2}) + P(|\tau_n - \tau| > \frac{\eta}{2}) \leq \frac{\epsilon}{2} + P(|\tau_n - \tau| > \frac{\eta}{2})$$

Since τ_n converges to τ , there exists some n_0 , such that for all $n \geq n_0$, $P(|\tau_n - \tau| > \frac{\eta}{2}) \leq \frac{\epsilon}{2}$. Hence $\tau_{n,\phi(n)} \xrightarrow{P} \tau$. So in order to prove the lemma, it only remains to prove the claim above.

Proof of the claim. We drop the index n and assume that τ is a \mathbb{G} stopping time that takes a finite number of values t_1, \dots, t_M . Since \mathbb{G} is right-continuous, $1_{\{\tau=t_i\}}$ is \mathcal{G}_{t_i} measurable, and since by assumption, for all i , $\mathcal{G}_{t_i}^m \xrightarrow{w} \mathcal{G}_{t_i}$, it follows that for all i

$$E(1_{\{\tau=t_i\}} | \mathcal{G}_{t_i}^m) \xrightarrow{P} 1_{\{\tau=t_i\}}$$

Now for $i = 1$, we can extract a subsequence $E(1_{\{\tau=t_1\}} | \mathcal{G}_{t_1}^{\phi_1(m)})$ converging to $1_{\{\tau=t_1\}}$ a.s. and any sub-subsequence will also converge to $1_{\{\tau=t_1\}}$ a.s. Also, $\mathcal{G}_{t_2}^{\phi_1(m)} \xrightarrow{w} \mathcal{G}_{t_2}$, hence $E(1_{\{\tau=t_2\}} | \mathcal{G}_{t_2}^{\phi_1(m)}) \xrightarrow{P} 1_{\{\tau=t_2\}}$, and we can extract a further subsequence $E(1_{\{\tau=t_2\}} | \mathcal{G}_{t_2}^{\phi_1(\phi_2(m))})$ that converges a.s. to $1_{\{\tau=t_2\}}$. Since we have a finite number of possible values, we can repeat this reasoning up to time t_M . Define then $\phi = \phi_1 \circ \phi_2 \circ \dots \circ \phi_n$, we get for all $i \in \{1, \dots, M\}$,

$$E(1_{\{\tau=t_i\}} | \mathcal{G}_{t_i}^{\phi(m)}) \xrightarrow{a.s.} 1_{\{\tau=t_i\}}.$$

Define $\tau_m = \min_{\{i | E(1_{\{\tau=t_i\}} | \mathcal{G}_{t_i}^m) > \frac{1}{2}\}} t_i$. Then

$$\{\tau_m = t_i\} = \{E(1_{\{\tau=t_i\}} | \mathcal{G}_{t_i}^m) > \frac{1}{2}\} \cap \{\forall t_j < t_i, E(1_{\{\tau=t_j\}} | \mathcal{G}_{t_j}^m) \leq \frac{1}{2}\}$$

and hence τ_m is a \mathbb{G}^m stopping time. Also, obviously, $\tau_{\phi(m)} \xrightarrow{a.s.} \tau$, hence $\tau_m \xrightarrow{P} \tau$. ■

2.3 Weak convergence of σ -fields and the semimartingale property

Assume we are given a sequence of filtrations $(\mathbb{F}^m)_{m \geq 1}$ and define the filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}$, where $\tilde{\mathcal{F}}_t = \bigvee_m \mathcal{F}_t^m$. We prove in this section a stability result for $\tilde{\mathbb{F}}$ semimartingales. More precisely, we prove that if X is an $\tilde{\mathbb{F}}$ semimartingale, then it remains an \mathbb{F} semimartingale for any limiting (in the sense $\mathcal{F}_t^m \xrightarrow{w} \mathcal{F}_t$, for all $t \in [0, T]$) filtration \mathbb{F} to which it is adapted.

The crucial tool for proving our first theorem is the Bichteler-Dellacherie characterization of semimartingales (see for example [15]). Recall that if \mathbb{H} is a filtration, an \mathbb{H} predictable elementary process H is a process of the form

$$H_t(\omega) = \sum_{i=1}^k h_i(\omega) 1_{[t_i, t_{i+1}]}(t);$$

where $0 \leq t_1 \leq \dots \leq t_{k+1} < \infty$, and each h_i is \mathcal{H}_{t_i} measurable. Moreover, for any \mathbb{H} adapted càdlàg process X and predictable elementary process H of the above form, we write

$$J_X(H) = \sum_{i=1}^k h_i(X_{t_{i+1}} - X_{t_i})$$

Theorem 1 (Bichteler-Dellacherie) *Let X be an \mathbb{H} adapted càdlàg process. Suppose that for every sequence $(H_n)_{n \geq 1}$ of bounded, \mathbb{H} predictable elementary processes that are null outside a fixed interval $[0, N]$ and convergent to zero uniformly in $(\omega; t)$, we have that $\lim_{n \rightarrow \infty} J_X(H_n) = 0$ in probability. Then X is an \mathbb{H} semimartingale.*

The converse is true by the Dominated Convergence Theorem for stochastic integrals. We can now state and prove the main theorem of this subsection.

Theorem 2 *Let $(\mathbb{F}^m)_{m \geq 1}$ be a sequence of filtrations. Let \mathbb{F} be a filtration such that for all $t \in [0, T]$, $\mathcal{F}_t^m \xrightarrow{w} \mathcal{F}_t$. Define the filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}$, where $\tilde{\mathcal{F}}_t = \bigvee_m \mathcal{F}_t^m$. Let X be an \mathbb{F} adapted càdlàg process such that X is an \mathbb{F} semimartingale. Then X is an $\tilde{\mathbb{F}}$ semimartingale.*

Proof. For a fixed $N > 0$, consider a sequence of bounded, \mathbb{F} predictable elementary processes of the form

$$H_t^n = \sum_{i=1}^{k_n} h_i^n 1_{[t_i^n, t_{i+1}^n]}(t);$$

null outside the fixed time interval $[0, N]$ and with h_i^n being $\mathcal{F}_{t_i^n}^n$ measurable. Suppose that H^n converges to zero uniformly in (ω, t) . We prove that $J_X(H^n) \xrightarrow{P} 0$.

For each m , define the sequence of bounded \mathbb{F}^m predictable elementary processes

$$H_t^{n,m} = \sum_{i=1}^{k_n} E(h_i^n \mid \mathcal{F}_{t_i^n}^m) 1_{[t_i^n, t_{i+1}^n]}(t);$$

By assumption, $\mathcal{F}_t^m \xrightarrow{w} \mathcal{F}_t$ for all $0 \leq t \leq T$. Hence for all n and $1 \leq i \leq k_n$, $\mathcal{F}_{t_i^n}^m \xrightarrow{w} \mathcal{F}_{t_i^n}$. Since h_i^n is bounded (hence integrable) and $\mathcal{F}_{t_i^n}$ measurable, it follows from Lemma 1 that $E(h_i^n | \mathcal{F}_{t_i^n}^m) \xrightarrow{P} h_i^n$ and hence $E(h_i^n | \mathcal{F}_{t_i^n}^m)(X_{t_{i+1}^n} - X_{t_i^n}) \xrightarrow{P} h_i^n(X_{t_{i+1}^n} - X_{t_i^n})$ for each n and $1 \leq i \leq k_n$ since $(X_{t_{i+1}^n} - X_{t_i^n})$ is finite a.s. Let $\eta > 0$.

$$P\left(|J_X(H^{n,m}) - J_X(H^n)| > \eta\right) \leq \sum_{i=1}^{k_n} P\left(\left|(E(h_i^n | \mathcal{F}_{t_i^n}^m) - h_i^n)(X_{t_{i+1}^n} - X_{t_i^n})\right| > \frac{\eta}{k_n}\right)$$

For each fixed n , the right side quantity converges to 0 as m tends to ∞ . This proves that for each n ,

$$J_X(H^{n,m}) \xrightarrow{P} J_X(H^n).$$

Let $\delta > 0$ and $\epsilon > 0$. For each n and m ,

$$P(|J_X(H^n)| > \delta) \leq P\left(|J_X(H^{n,m}) - J_X(H^n)| > \frac{\delta}{2}\right) + P(|J_X(H^{n,m})| > \frac{\delta}{2}) \quad (2)$$

From $J_X(H^{n,m}) \xrightarrow{P} J_X(H^n)$, it follows that for each n , there exists M_0^n such that for all $m \geq M_0^n$,

$$P\left(|J_X(H^{n,m}) - J_X(H^n)| > \frac{\delta}{2}\right) \leq \frac{\epsilon}{2}$$

Hence $P(|J_X(H^n)| > \delta) \leq \frac{\epsilon}{2} + P(|J_X(H^{n,M_0^n})| > \frac{\delta}{2})$. First $E(h_i^n | \mathcal{F}_{t_i^n}^{M_0^n})$ is bounded, $\tilde{\mathcal{F}}_{t_i^n}^{M_0^n}$ measurable so that $H_t^{n,M_0^n} = \sum_{i=1}^{k_n} E(h_i^n | \mathcal{F}_{t_i^n}^{M_0^n}) 1_{[t_i^n, t_{i+1}^n]}(t)$ is a bounded $\tilde{\mathbb{F}}$ predictable process. Since H^n converges to zero uniformly in (ω, t) , it follows that h_i^n converges to zero uniformly in (ω, i) so that there exists n_0 such that for each $n \geq n_0$, for all (ω, i) , $|h_i^n(\omega)| \leq \epsilon$. Hence, for all (ω, t) and $n \geq n_0$

$$|H_t^{n,M_0^n}(\omega)| \leq \sum_{i=1}^{k_n} E(|h_i^n| | \mathcal{F}_{t_i^n}^{M_0^n})(\omega) 1_{[t_i^n, t_{i+1}^n]}(t) \leq \epsilon \sum_{i=1}^{k_n} 1_{[t_i^n, t_{i+1}^n]}(t) \leq \epsilon$$

Therefore H^{n,M_0^n} is a sequence of bounded $\tilde{\mathbb{F}}$ predictable processes null outside the fixed interval $[0, N]$ that converges uniformly to zero in (ω, t) . Since by assumption X is a $\tilde{\mathbb{F}}$ semimartingale, it follows from the converse of Bichteler-Dellacherie's theorem that $J_X(H^{n,M_0^n})$ converges to zero in probability, hence, for n large enough, $P(|J_X(H^{n,M_0^n})| > \frac{\delta}{2}) \leq \frac{\epsilon}{2}$ and

$$P(|J_X(H^n)| > \delta) \leq \epsilon$$

Applying now Theorem 1 proves that X is an \mathbb{F} semimartingale. ■

Let X be an $\tilde{\mathbb{F}}$ semimartingale. Theorem 2 proves that X remains an \mathbb{F} semimartingale for any limiting filtration \mathbb{F} (in the sense $\mathcal{F}_t^m \xrightarrow{w} \mathcal{F}_t$ for all $0 \leq t \leq T$) to which X is adapted. Of course, if $\mathbb{F} \subset \tilde{\mathbb{F}}$, Stricker's theorem already implies that X is an \mathbb{F} semimartingale. But there is no general link between the filtration $\tilde{\mathbb{F}} = \bigvee_m \mathbb{F}^m$ and the limiting filtration \mathbb{F} . A trivial example is given by taking \mathbb{F} to be the trivial filtration (it can be seen from Definition 1 that the trivial filtration satisfies $\mathcal{F}_t^m \xrightarrow{w} \mathcal{F}_t$, for all t , for any given sequence of filtrations \mathbb{F}^m). One can also have $\bigvee_m \mathbb{F}^m \subset \mathbb{F}$, as it is the case in the following important example.

Example 1 Let X be a càdlàg process. Consider a sequence of subdivisions $\{t_k^n\}$ whose mesh tends to zero and let X^n be the discretized process defined by $X_t^n = X_{t_k^n}$, for all $t_k^n \leq t < t_{k+1}^n$. Let \mathbb{F} and \mathbb{F}^n be the natural filtrations of X and X^n . It is well known that for all t , $\mathcal{F}_{t-} \subset \bigvee_n \mathcal{F}_t^n \subset \mathcal{F}_t$. Also, X^n converges a.s. to the process X , hence $X^n \xrightarrow{P} X$. Assume now that X has no fixed times of discontinuity. Then Lemma 4 guarantees that $\mathcal{F}_t^n \xrightarrow{w} \mathcal{F}_t$, for all t . Moreover, if \mathbb{F} is left-continuous (which is usually the case, and holds for example when X is a càdlàg Hunt Markov process) then $\bigvee_n \mathcal{F}_t^n = \mathcal{F}_t$ for all t .

We provide now another example where $\bigvee_n \mathcal{F}_t^n$ is itself a limiting σ -field for $(\mathcal{F}_t^n)_{n \geq 1}$, for each t .

Example 2 Assume that \mathbb{F}^n is a sequence of filtrations such that for all t , the sequence of σ -fields $(\mathcal{F}_t^n)_{n \geq 1}$ is increasing for the inclusion. Define $\tilde{\mathcal{F}}_t = \bigvee_n \mathcal{F}_t^n$. Then for each t , $\mathcal{F}_t^n \xrightarrow{w} \tilde{\mathcal{F}}_t$. To see this, fix t and let X be an integrable $\tilde{\mathcal{F}}_t$ measurable random variable. Then $M_n = E(X | \mathcal{F}_t^n)$ is a closed martingale and the convergence theorem for closed martingales ensures that M_n converges to X in L^1 , which implies that $E(X | \mathcal{F}_t^n) \xrightarrow{P} X$. Lemma 1 allows us to conclude.

Checking in practice that X is an $\tilde{\mathbb{F}}$ semimartingale can be a hard task. In subsequent sections, we replace the strong assumption *X is an $\tilde{\mathbb{F}}$ semimartingale* by the more natural assumption *X is an \mathbb{F}^n semimartingale, for each n* . Theorem 2 is very instructive since we see from the proof what goes wrong under this new assumption : the change in the order of limits in (2) cannot be justified anymore and extra integrability conditions will be needed. They are introduced in the next section.

This assumption arises naturally in filtration expansion theory in the following way. Assume we are given a base filtration \mathbb{F} and a sequence of processes N^n which converges (in probability for the Skorohod J_1 topology) to some process N . Let \mathbb{N}^n and \mathbb{N} be their natural filtrations and \mathbb{G}^n (resp. \mathbb{G}) the smallest right-continuous filtration containing \mathbb{F} and to which N^n (resp. N) is adapted. Assume that for each n , every \mathbb{F} semimartingale remains a \mathbb{G}^n semimartingale. Does this property also hold between \mathbb{F} and \mathbb{G} ? In the next section we answer this question under the assumption of weak convergence of the σ -fields \mathcal{G}_t^n to \mathcal{G}_t for each t , for a class of \mathbb{F} semimartingales X satisfying some integrability conditions. If moreover $\mathbb{G}^n \xrightarrow{w} \mathbb{G}$, we are able to provide the \mathbb{G} decomposition of such X .

3 Filtration expansion with processes

In preparation for treating the expansion of filtrations via processes, we need to establish a general result on the convergence of semimartingales, which is perhaps of interest in its own right.

3.1 Convergence of semimartingales

The following theorem is a generalization of the main result in [3].

Theorem 3 Let $(\mathbb{G}^n)_{n \geq 1}$ be a sequence of right-continuous filtrations and let \mathbb{G} be a filtration such that $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$ for all t . Let $(X^n)_{n \geq 1}$ be a sequence of \mathbb{G}^n semimartingales with canonical decomposition $X^n = X_0^n + M^n + A^n$. Assume there exists $K > 0$ such that for all n ,

$$E\left(\int_0^T |dA_s^n|\right) \leq K \quad \text{and} \quad E\left(\sup_{0 \leq s \leq T} |M_s^n|\right) \leq K$$

Then the following holds.

- (i) Assume there exists a \mathbb{G} adapted process X such that $E(\sup_{0 \leq s \leq T} |X_s^n - X_s|) \rightarrow 0$. Then X is a \mathbb{G} special semimartingale.
- (ii) Moreover, assume \mathbb{G} is right-continuous and let $X = M + A$ be the canonical decomposition of X . Then M is a \mathbb{G} martingale and $\int_0^T |dA_s|$ and $\sup_{0 \leq s \leq T} |M_s|$ are integrable.

Proof. Part (i). The idea of the proof of Part (i) is similar to the one in [3]. First, X is càdlàg since it is the a.s. uniform limit of a subsequence of the càdlàg processes $(X^n)_{n \geq 1}$. Also, since $\|X_0^n - X_0\|_1 \rightarrow 0$, we can take w.l.o.g $X_0^n = X_0 = 0$, and we do so. The integrability assumptions guarantee that $E(\sup_s |X_s^n|) \leq 2K$ and up to replacing K by $2K$, we assume that $E(\sup_s |M_s^n|) \leq K$, $E(\int_0^T |dA_s^n|) \leq K$ and $E(\sup_s |X_s^n|) \leq K$. Then $E(\sup_s |X_s|) \leq E(\sup_s |X_s - X_s^n|) + K$ and by taking limits $E(\sup_s |X_s|) \leq K$.

Let H be a \mathbb{G} predictable elementary process of the form $H_t = \sum_{i=1}^k h_i 1_{[t_i, t_{i+1}]}(t)$, where h_i is a \mathcal{G}_{t_i} measurable random variable such that $|h_i| \leq 1$ and $t_1 < \dots < t_k < t_{k+1} = T$. Define now $H_t^n = \sum_{i=1}^k h_i^n 1_{[t_i, t_{i+1}]}(t)$, where $h_i^n = E(h_i | \mathcal{G}_{t_i}^n)$. Then h_i^n is a $\mathcal{G}_{t_i}^n$ measurable random variable satisfying $|h_i^n| \leq 1$, hence H^n is a bounded \mathbb{G}^n predictable elementary process. It follows that $H^n \cdot M^n$ is a \mathbb{G}^n martingale and for each n ,

$$|E((H^n \cdot X^n)_T)| \leq |E(\int_0^T H_s^n dA_s^n)| \leq E(\int_0^T |dA_s^n|) \leq K$$

Therefore, for each n ,

$$|E((H \cdot X)_T)| \leq |E((H \cdot X)_T - (H^n \cdot X^n)_T)| + K \tag{3}$$

Since h_i is \mathcal{G}_{t_i} measurable and $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$ for all t , $h_i^n \xrightarrow{P} h_i$ for all $1 \leq i \leq k$. Since the set $\{1, \dots, k\}$ is finite, successive extractions allow us to find a subsequence $\psi(n)$ (independent from i) such that for each $1 \leq i \leq k$, $h_i^{\psi(n)}$ converges a.s. to h_i . So up to working with the $\mathbb{G}^{\psi(n)}$ predictable elementary processes $H^{\psi(n)}$ and the stochastic integrals $H^{\psi(n)} \cdot X^{\psi(n)}$ in (3), we can assume that h_i^n converges a.s. to h_i , for each $1 \leq i \leq k$.

Now, $|E((H \cdot X)_T - (H^n \cdot X^n)_T)| \leq \sum_{i=1}^k E(|h_i Y_i - h_i^n Y_i^n|)$ where $Y_i = X_{t_{i+1}} - X_{t_i}$ and $Y_i^n = X_{t_{i+1}}^n - X_{t_i}^n$. Each term in the sum can be bounded as follows.

$$\begin{aligned}
E(|h_i Y_i - h_i^n Y_i^n|) &\leq E(|Y_i^n(h_i^n - h_i)|) + E(|h_i(Y_i^n - Y_i)|) \\
&\leq 2E(\sup_s |X_s^n| |h_i^n - h_i|) + E(|Y_i^n - Y_i|) \\
&\leq 2E(\sup_s |X_s^n - X_s| |h_i^n - h_i|) + 2E(\sup_s |X_s| |h_i^n - h_i|) + 2E(\sup_s |X_s^n - X_s|) \\
&\leq 6E(\sup_s |X_s^n - X_s|) + 2E(\sup_s |X_s| |h_i^n - h_i|)
\end{aligned}$$

Since $\sup_s |X_s| |h_i^n - h_i|$ converges a.s to zero and that for all n , $|h_i^n| \leq 1$, hence $\sup_s |X_s| |h_i^n - h_i| \leq 2 \sup_s |X_s|$ and $\sup_s |X_s| \in L^1$, the Dominated Convergence Theorem implies that $E(\sup_s |X_s| |h_i^n - h_i|) \rightarrow 0$. Since by assumption $E(\sup_s |X_s^n - X_s|) \rightarrow 0$, it follows that $|E((H \cdot X)_T - (H^n \cdot X^n)_T)|$ converges to 0. Letting n tend to infinity in (3) gives $|E((H \cdot X)_T)| \leq K$. So X is a \mathbb{G} quasimartingale, hence a \mathbb{G} special semimartingale. Therefore X has a \mathbb{G} canonical decomposition $X = M + A$ where M is a \mathbb{G} local martingale and A is a \mathbb{G} predictable finite variation process.

Part(ii). Let $(\tau_m)_{m \geq 1}$ be a sequence of bounded \mathbb{G} stopping times that reduces M . Since for all t , $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$, it follows from Lemma 8 that for each m there exist a function ϕ_m strictly increasing and a sequence $(\tau_m^n)_{n \geq 1}$ such that $(\tau_m^{\phi_m(n)})_{n \geq 1}$ converges in probability to τ_m and $\tau_m^{\phi_m(n)}$ are bounded $\mathbb{G}^{\phi_m(n)}$ stopping times. We can extract a subsequence $(\tau_m^{\phi_m(\psi_m(n))})_{n \geq 1}$ converging a.s. to τ_m . In order to simplify the notation, fix $m \geq 1$ and up to working with $\tilde{\mathbb{G}}^n = \mathbb{G}^{\phi_m(\psi_m(n))}$ instead of \mathbb{G}^n (which satisfies the same assumptions), take $\Phi_m := \phi_m \circ \psi_m$ to be the identity. Let H be a \mathbb{G} elementary predictable process as defined in Part (i). Since τ_m reduces M , $E((H \cdot A)_{\tau_m}) = E((H \cdot X)_{\tau_m})$. We can write

$$E((H \cdot A)_{\tau_m}) = E((H \cdot X)_{\tau_m} - (H^n \cdot X^n)_{\tau_m}) + E((H^n \cdot X^n)_{\tau_m} - (H^n \cdot X^n)_{\tau_m^n}) + E((H^n \cdot X^n)_{\tau_m^n})$$

We start with the third term. Since $H^n \cdot M^n$ is a \mathbb{G}^n martingale and τ_m^n is a bounded \mathbb{G}^n stopping time, it follows from Doob's optional sampling theorem that $E((H^n \cdot X^n)_{\tau_m^n}) = E((H^n \cdot A^n)_{\tau_m^n})$, hence $|E((H^n \cdot X^n)_{\tau_m^n})| \leq E(\int_0^{\tau_m^n} |dA_s^n|) \leq K$.

We focus now on the first term. Let $Y_s^i = X_s - X_{t_i}$ and $Y_s^{i,n} = X_s^n - X_{t_i}^n$.

$$\begin{aligned}
E_1 &:= |E((H \cdot X)_{\tau_m} - (H^n \cdot X^n)_{\tau_m})| \leq E\left(\sup_{0 \leq s \leq T} |(H \cdot X)_s - (H^n \cdot X^n)_s|\right) \\
&\leq E\left(\sum_{i=1}^k \sup_{t_i < s \leq t_{i+1}} |h_i Y_s^i - h_i^n Y_s^{i,n}|\right) \\
&\leq \sum_{i=1}^k E\left(\sup_{t_i < s \leq t_{i+1}} |Y_s^{i,n}| |h_i^n - h_i| + \sup_{t_i < s \leq t_{i+1}} |h_i| |Y_s^{i,n} - Y_s^i|\right)
\end{aligned}$$

Since $|h^i| \leq 1$, $|Y_s^{i,n}| \leq 2 \sup_u |X_u^n|$ and $|Y_s^{i,n} - Y_s^i| \leq 2 \sup_u |X_u^n - X_u|$, it follows that

$$\begin{aligned} E_1 &\leq 2 \sum_{i=1}^k \left\{ E \left(\sup_{0 \leq u \leq T} |X_s^n| |h_i^n - h_i| \right) + E \left(\sup_{0 \leq u \leq T} |X_u^n - X_u| \right) \right\} \\ &\leq 6kE \left(\sup_{0 \leq s \leq T} |X_s^n - X_s| \right) + 2 \sum_{i=1}^k \left\{ E \left(\sup_{0 \leq s \leq T} |X_s| |h_i^n - h_i| \right) \right\} \end{aligned}$$

We study now the second term $E_2 := E((H^n \cdot X^n)_{\tau_m} - (H^n \cdot X^n)_{\tau_m^n})$. Let $0 < \eta < \min_i |t_{i+1} - t_i|$, and define $Y^n = H^n \cdot X^n$. Write now

$$E(|Y_{\tau_m}^n - Y_{\tau_m^n}^n|) = E(|Y_{\tau_m}^n - Y_{\tau_m^n}^n| 1_{\{|\tau_m - \tau_m^n| \leq \eta\}}) + E(|Y_{\tau_m}^n - Y_{\tau_m^n}^n| 1_{\{|\tau_m - \tau_m^n| > \eta\}}) =: e_1 + e_2$$

We study each of the two terms separately. We start with e_2 .

$$\begin{aligned} e_2 &\leq E((|Y_{\tau_m}^n| + |Y_{\tau_m^n}^n|) 1_{\{|\tau_m - \tau_m^n| > \eta\}}) \leq 2E \left(\sum_{i=1}^k \sup_{t_i < s \leq t_{i+1}} |X_s^n - X_{t_i}^n| 1_{\{|\tau_m - \tau_m^n| > \eta\}} \right) \\ &\leq 4kE \left(\sup_{0 \leq u \leq T} |X_u^n| 1_{\{|\tau_m - \tau_m^n| > \eta\}} \right) \\ &\leq 4kE \left(\sup_{0 \leq u \leq T} |X_u^n - X_u| \right) + 4kE \left(\sup_{0 \leq u \leq T} |X_u| 1_{\{|\tau_m - \tau_m^n| > \eta\}} \right) \end{aligned}$$

We study now e_1 . On $\{|\tau_m - \tau_m^n| \leq \eta\}$ and since $\eta < \min_i |t_{i+1} - t_i|$, we have $|Y_{\tau_m}^n - Y_{\tau_m^n}^n| \leq 2 \sup_{s \leq t \leq s+\eta} |X_t^n - X_s^n|$. In fact, one of the two following cases is possible for τ_m and τ_m^n . Either they are both in the same interval $(t_i, t_{i+1}]$, in which case, $|Y_{\tau_m}^n - Y_{\tau_m^n}^n| = |h_i^n(X_{\tau_m}^n - X_{\tau_m^n}^n)| \leq \sup_{s \leq t \leq s+\eta} |X_t^n - X_s^n|$, or they are in two consecutive intervals. For the second case, take for example $t_{i-1} < \tau_m \leq t_i < \tau_m^n \leq t_{i+1}$, then

$$\begin{aligned} |Y_{\tau_m}^n - Y_{\tau_m^n}^n| &= |h_{i-1}^n(X_{\tau_m}^n - X_{t_{i-1}}^n) - h_{i-1}^n(X_{t_i}^n - X_{t_{i-1}}^n) - h_i^n(X_{\tau_m^n}^n - X_{t_i}^n)| \\ &= |h_{i-1}^n(X_{\tau_m}^n - X_{t_i}^n) - h_i^n(X_{\tau_m^n}^n - X_{t_i}^n)| \leq |X_{\tau_m}^n - X_{t_i}^n| + |X_{\tau_m^n}^n - X_{t_i}^n| \\ &\leq 2 \sup_{s \leq t \leq s+\eta} |X_t^n - X_s^n| \end{aligned}$$

The case $t_{i-1} < \tau_m^n \leq t_i < \tau_m \leq t_{i+1}$ is similar. Hence

$$e_1 \leq 2E \left(\sup_{s \leq t \leq s+\eta} |X_t^n - X_s^n| 1_{\{|\tau_m - \tau_m^n| \leq \eta\}} \right) \leq 2E \left(\sup_{s \leq t \leq s+\eta} |X_t^n - X_s^n| \right)$$

Putting all this together yields for each $0 < \eta < \min_i |t_{i+1} - t_i|$ and each n

$$\begin{aligned} |E(H \cdot A)_{\tau_m}| &\leq K + 2E \left(\sup_{s \leq t \leq s+\eta} |X_t^n - X_s^n| \right) + 2 \sum_{i=1}^k E \left(\sup_{0 \leq s \leq T} |X_s| |h_i^n - h_i| \right) \\ &\quad + 4kE \left(\sup_{0 \leq s \leq T} |X_s| 1_{\{|\tau_m - \tau_m^n| > \eta\}} \right) + 10kE \left(\sup_{0 \leq u \leq T} |X_u^n - X_u| \right) \end{aligned}$$

Getting back to the general case, we obtain for each $m \geq 1$ and each $n \geq 1$,

$$|E(H \cdot A)_{\tau_m}| \leq K + 2E\left(\sup_{s \leq t \leq s+\eta} |X_t^{\Phi_m(n)} - X_s^{\Phi_m(n)}|\right) + 2 \sum_{i=1}^k E\left(\sup_{0 \leq s \leq T} |X_s| |h_i^{\Phi_m(n)} - h_i|\right) \\ + 4kE\left(\sup_{0 \leq s \leq T} |X_s| 1_{\{|\tau_m - \tau_m^{\Phi_m(n)}| > \eta\}}\right) + 10kE\left(\sup_{0 \leq u \leq T} |X_u^{\Phi_m(n)} - X_u|\right)$$

As in the proof of Part (i), successive extractions allow us to find $\lambda_m(n)$ (independent from i) such that for all $1 \leq i \leq k$, $h_i^{\lambda_m(n)}$ converges a.s. to h_i . Letting n go to infinity in

$$|E(H \cdot A)_{\tau_m}| \leq K + 2E\left(\sup_{s \leq t \leq s+\eta} |X_t^{\lambda_m(n)} - X_s^{\lambda_m(n)}|\right) + 2 \sum_{i=1}^k E\left(\sup_{0 \leq s \leq T} |X_s| |h_i^{\lambda_m(n)} - h_i|\right) \\ + 4kE\left(\sup_{0 \leq s \leq T} |X_s| 1_{\{|\tau_m - \tau_m^{\lambda_m(n)}| > \eta\}}\right) + 10kE\left(\sup_{0 \leq u \leq T} |X_u^{\lambda_m(n)} - X_u|\right)$$

gives the estimate

$$|E(H \cdot A)_{\tau_m}| \leq K + 2 \limsup_{n \rightarrow \infty} E\left(\sup_{s \leq t \leq s+\eta} |X_t^n - X_s^n|\right)$$

Let $\lim_{\eta \rightarrow 0} \limsup_{n \rightarrow \infty} E(\sup_{s \leq t \leq s+\eta} |X_t^n - X_s^n|) = C$. Since $E(\sup_{s \leq t \leq s+\eta} |X_t^n - X_s^n|) \leq 2E(\sup_u |X_u^n|) \leq 2(E(\sup_u |M_u^n| + \int_0^T |dA_s^n|)) \leq 4K$, $C < \infty$. Now letting η go to zero yields finally $|E(H \cdot A)_{\tau_m}| \leq K + 2C$, for each m . Thus $E(\int_0^{\tau_m} |dA_s|) \leq K + 2C$, for each m and hence $E(\int_0^T |dA_s|) \leq K + 2C$.

Now, $M = X - A = (X - X^n) + M^n + A^n - A$, and so

$$\sup_{0 \leq s \leq T} |M_s| \leq \sup_{0 \leq s \leq T} |X_s - X_s^n| + \sup_{0 \leq s \leq T} |M_s^n| + \int_0^T |dA_s^n| + \int_0^T |dA_s|$$

Thus $E(\sup_{0 \leq s \leq T} |M_s|) \leq 3K + 2C$ and M is a \mathbb{G} martingale. ■

Once one obtains that X is a \mathbb{G} special semimartingale, one can be interested in characterizing the martingale M and the finite variation predictable process A in terms of the processes M^n and A^n . Mémin (Theorem 11 in [14]) achieved this under “extended convergence.” Recall that (X^n, \mathbb{G}^n) converges to (X, \mathbb{G}) in the extended sense if for every $G \in \mathbb{G}_T$, the sequence of càdlàg processes $(X_t^n, E(1_G | \mathcal{G}_t^n))_{0 \leq t \leq T}$ converges in probability under the Skorohod J_1 topology to $(X_t, E(1_G | \mathcal{G}_t))_{0 \leq t \leq T}$. The author proves the following theorem. We refer to [14] for a proof. In the theorem below, \mathbb{G}^n and \mathbb{G} are right-continuous filtrations.

Theorem 4 *Let $(X^n)_{n \geq 1}$ be a sequence of \mathbb{G}^n special semimartingales with canonical decompositions $X^n = M^n + A^n$ where M^n is a \mathbb{G}^n martingale and A^n is a \mathbb{G}^n predictable finite variation process. We suppose that the sequence $([X^n, X^n]_T^{\frac{1}{2}})_{n \geq 1}$ is uniformly integrable and that the sequence $(V(A^n)_T)_{n \geq 1}$ (where V denotes the variation process) of real random variables is tight in \mathbb{R} . Let X be a \mathbb{G} quasi-left continuous special semimartingale with a canonical decomposition $X = M + A$ such that $([X, X]_T^{\frac{1}{2}}) < \infty$.*

If the extended convergence $(X^n, \mathbb{G}^n) \rightarrow (X, \mathbb{G})$ holds, then (X^n, M^n, A^n) converges in probability under the Skorohod J_1 topology to (X, M, A) .

In a filtration expansion setting, the sequence X^n is constant and equal to some semimartingale X of the base filtration. In this case the extended convergence assumption in Theorem 4 reduces to the weak convergence of the filtrations. We can deduce the following corollary from Theorems 3 and 4.

Corollary 1 *Let $(\mathbb{G}^n)_{n \geq 1}$ be a sequence of right-continuous filtrations and let \mathbb{G} be a filtration such that $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$ for all t . Let X be a stochastic process such that for each n , X is a \mathbb{G}^n semimartingale with canonical decomposition $X = M^n + A^n$ such that there exists $K > 0$, $E(\int_0^T |dA_s^n|) \leq K$ and $E(\sup_{0 \leq s \leq T} |M_s^n|) \leq K$ for all n . Then*

- (i) *If X is \mathbb{G} adapted, then X is a \mathbb{G} special semimartingale.*
- (ii) *Assume moreover that \mathbb{G} is right-continuous and let $X = M + A$ be the canonical decomposition of X . Then M is a \mathbb{G} martingale and $\sup_{0 \leq s \leq T} |M_s|$ and $\int_0^T |dA_s|$ are integrable.*
- (iii) *Furthermore, assume that X is \mathbb{G} quasi-left continuous and $\mathbb{G}^n \xrightarrow{w} \mathbb{G}$. Then (M^n, A^n) converges in probability under the Skorohod J_1 topology to (M, A) .*

Proof. The sequence $X^n = X$ clearly satisfies the assumptions of Theorem 3, and the two first claims follow. For the last claim, notice that $[X, X]_T \in L^1$, so $\sqrt{[X, X]_T} \in L^1$ and hence $(\sqrt{[X, X]_t})_{0 \leq t \leq T}$ is a uniformly integrable family of random variables. The tightness of the sequence of random variables $(V(A^n)_T)_{n \geq 1}$ follows from $E(\int_0^T |dA_s^n|) \leq K$ for any n and some K independent from n . ■

3.2 Applications to filtration expansions

We provide in this subsection a first application to the initial and progressive filtration expansions with a random variable and a general theorem on the progressive expansion with a process. We assume in the sequel that a right-continuous filtration \mathbb{F} is given.

3.2.1 Initial and progressive filtration expansions with a random variable

Assume that \mathbb{F} is the natural filtration of some càdlàg process. Let τ be a random variable and \mathbb{H} and \mathbb{G} the initial and progressive expansions of \mathbb{F} with τ . In this subsection, the filtration \mathbb{G} is considered only when τ is non negative. It is proved in [13] that if τ satisfies Jacod's criterion i.e. if there exists a σ -finite measure η on $\mathcal{B}(\mathbb{R})$ such that

$$P(\tau \in \cdot \mid \mathcal{F}_t)(\omega) \ll \eta(\cdot) \quad \text{a.s.}$$

then every \mathbb{F} semimartingale remains an \mathbb{H} and \mathbb{G} semimartingale. That it is an \mathbb{H} semimartingale is due to Jacod [10]. That it is also a \mathbb{G} semimartingale follows from Stricker's theorem. Its \mathbb{G} decomposition is obtained in [13] and this relies on the fact that these two filtrations *coincide* after τ . We provide now a similar but partial result for a random

variable τ which may not satisfy Jacod's criterion. Assume there exists a sequence of random times $(\tau_n)_{n \geq 0}$ converging in probability to τ and let \mathbb{H}^n be the initial expansion of \mathbb{F} with τ_n . The following holds.

Theorem 5 *Let M be an \mathbb{F} martingale such that $\sup_{0 \leq t \leq T} |M_t|$ is integrable. Assume there exists an \mathbb{H}^n predictable finite variation process A^n such that $M - A^n$ is an \mathbb{H}^n martingale. If there exists K such that $E(\int_0^T |dA_s^n|) \leq K$ for all n , then M is an \mathbb{H} and \mathbb{G} semimartingale.*

Proof. Since τ_n converges in probability to τ and \mathbb{F} is the natural filtration of some càdlàg process, we can prove that $\mathcal{H}_t^n \xrightarrow{w} \mathcal{H}_t$ for each $t \in [0, T]$, using the same techniques as in Lemmas 3 and 4. Up to replacing K by $K + E(\sup_{0 \leq t \leq T} |M_t|)$, $M^n = M - A^n$ and A^n satisfy the assumptions of Corollary 1. Therefore M is an \mathbb{H} semimartingale, and a \mathbb{G} semimartingale by Stricker's theorem. ■

One case where the first assumption of Theorem 5 is satisfied is when τ_n satisfies Jacod's criterion, for each $n \geq 0$. In this case, and if \mathbb{F} is the natural filtration of a Brownian motion W , the result above can be made more explicit. Assume for simplicity that the conditional distributions of τ_n are absolutely continuous w.r.t Lebesgue measure,

$$P(\tau_n \in du \mid \mathcal{F}_t)(\omega) = p_t^n(u, \omega) du$$

where the conditional densities are chosen so that $(u, \omega, t) \rightarrow p_t^n(u, \omega)$ is càdlàg in t and measurable for the optional σ -field associated with the filtration $\hat{\mathbb{F}}$ given by $\hat{\mathcal{F}}_t = \cap_{u>t} \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_u$. From the martingale representation theorem in a Brownian filtration, there exists for each n a family $\{q^n(u), u > 0\}$ of \mathbb{F} predictable processes $(q_t^n(u))_{0 \leq t \leq T}$ such that

$$p_t^n(u) = p_0^n(u) + \int_0^t q_s^n(u) dW_s \quad (4)$$

Corollary 2 *Assume there exists K such that $E\left(\int_0^T \left|\frac{q_s^n(\tau_n)}{p_s^n(\tau_n)}\right| ds\right) \leq K$ for all n , then W is a special semimartingale in both \mathbb{H} and \mathbb{G} .*

Proof. Since τ_n satisfy Jacod's criterion, it follows from Theorem 2.1 in [10] that $W_t - A_t^n$ is an \mathbb{H} local martingale, where $A_t^n = \int_0^t \frac{d\langle p^n(u), W \rangle_s}{p_s^n(u)} \Big|_{u=\tau_n}$. Now, it follows from (4) that $A_t^n = \int_0^t \frac{q_s^n(\tau_n)}{p_s^n(\tau_n)} ds$ and Theorem 5 allows us to conclude. ■

Assume the assumptions of Corollary 2 are satisfied and let $W = M + A$ be the \mathbb{H} canonical decomposition of W . Let m be an \mathbb{F} predictable process such that $\int_0^t m_s^2 ds$ is locally integrable and let M be the \mathbb{F} local martingale $M_t = \int_0^t m_s dW_s$. Theorem VI.5 in [15] then guarantees that M is an \mathbb{H} semimartingale as soon as the process $(\int_0^t m_s dA_s)_{t \geq 0}$ exists as a path-by-path Lebesgue-Stieltjes integral a.s.

Example 3 *In order to emphasize that some assumptions as in Theorem 5 are needed, we provide now a counter-example. Let \mathbb{F} be the natural filtration of some Brownian motion B and choose τ to be some functional of the Brownian path i.e. $\tau = f((B_s, 0 \leq s \leq 1))$, such that $\sigma(\tau) = \mathcal{F}_1$. Then B is not a semimartingale in $\mathbb{H} = (\mathcal{F}_t \vee \sigma(\tau))_{0 \leq t \leq 1}$. Now, define $\tau_n = \tau + \frac{1}{\sqrt{n}}N$, where N is a standard normal random variable independent from \mathbb{F} .*

Then τ_n converge a.s. to τ and $P(\tau_n \leq u \mid \mathcal{F}_t) = \int_{-\infty}^u E(g_n(v - \tau) \mid \mathcal{F}_t) dv$ where g_n is the probability density function of $\frac{1}{\sqrt{n}}N$, hence $P(\tau_n \in du \mid \mathcal{F}_t)(\omega) = E(g_n(u - \tau) \mid \mathcal{F}_t)(\omega)(du)$. Therefore, τ_n satisfies Jacod's criterion, for each n and $p_t^n(u, \omega) = E(g_n(u - \tau) \mid \mathcal{F}_t)(\omega)$. Thus, B is a semimartingale in $\mathbb{H}^n = (\mathcal{F}_t \vee \sigma(\tau_n)_{0 \leq t \leq 1})$ and $\mathcal{H}_t^n \xrightarrow{w} \mathcal{H}_t$ for each $0 \leq t \leq 1$.

3.2.2 Progressive filtration expansion with a process

Let $(N^n)_{n \geq 1}$ be a sequence of càdlàg processes converging in probability under the Skorohod J_1 -topology to a càdlàg process N and let \mathbb{N}^n and \mathbb{N} be their natural filtrations. Define the filtrations $\mathbb{G}^{0,n} = \mathbb{F} \vee \mathbb{N}^n$ and \mathbb{G}^n by $\mathcal{G}_t^n = \bigcap_{u > t} \mathcal{G}_u^{0,n}$. Let also \mathbb{G}^0 (resp. \mathbb{G}) be the smallest (resp. the smallest right-continuous) filtration containing \mathbb{F} and to which N is adapted. The result below is the main theorem of this paper.

Theorem 6 *Let X be an \mathbb{F} semimartingale such that for each n , X is a \mathbb{G}^n semimartingale with canonical decomposition $X = M^n + A^n$. Assume $E(\int_0^T |dA_s^n|) \leq K$ and $E(\sup_{0 \leq s \leq T} |M_s^n|) \leq K$ for some K and all n . Finally, assume one of the following holds.*

- N has no fixed times of discontinuity,
- N^n is a discretization of N along some refining subdivision $(\pi_n)_{n \geq 1}$ such that each fixed time of discontinuity of N belongs to $\cup_n \pi_n$.

Then

- (i) X is a \mathbb{G}^0 special semimartingale.
- (ii) Moreover, if \mathbb{F} is the natural filtration of some càdlàg process then X is a \mathbb{G} special semimartingale with canonical decomposition $X = M + A$ such that M is a \mathbb{G} martingale and $\sup_{0 \leq s \leq T} |M_s|$ and $\int_0^T |dA_s|$ are integrable.
- (iii) Furthermore, assume that X is \mathbb{G} quasi-left continuous and $\mathbb{G}^n \xrightarrow{w} \mathbb{G}$. Then (M^n, A^n) converges in probability under the Skorohod J_1 topology to (M, A) .

Proof. Under assumption (i), since $N^n \xrightarrow{P} N$ and $P(\Delta N_t \neq 0) = 0$ for all t , it follows from Lemma 4 that $\mathcal{N}_t^n \xrightarrow{P} \mathcal{N}_t$ for all t . The same holds under assumption (ii) using Lemma 5. Lemma 6 then ensures that $\mathcal{G}_t^{0,n} \xrightarrow{w} \mathcal{G}_t^0$ for all t . Since $\mathcal{G}_t^{0,n} \subset \bigcap_{u > t} \mathcal{G}_u^{0,n} = \mathcal{G}_t^n$, it follows from Lemma 7 that $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t^0$ for all t . Being an \mathbb{F} semimartingale, X is clearly \mathbb{G}^0 adapted. An application of Corollary 1 ends the proof of the first claim. When \mathbb{F} is the natural filtration of some càdlàg process, the same proofs as of Lemmas 4 and 5 guarantee that $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$ for all t . Since \mathbb{G} is right-continuous, the second and third claims follow from Corollary 1. ■

We apply this result to expand the filtration \mathbb{F} progressively with a point process. Let $(\tau_i)_{i \geq 1}$ and $(X_i)_{i \geq 1}$ be two sequences of random variables such that for each n , the random vector $(\tau_1, X_1, \dots, \tau_n, X_n)$ satisfies Jacod's criterion w.r.t the filtration \mathbb{F} . Assume that for all t and i , $P(\tau_i = t) = 0$ and that one of the following holds:

- (i) For all i , X_i and τ_i are independent, $E|X_i| = \mu$ for some μ and $\sum_{i=1}^{\infty} P(\tau_i \leq T) < \infty$

(ii) $E(|X_i^2|) = c$ and $\sum_{i=1}^{\infty} \sqrt{P(\tau_i \leq T)} < \infty$.

Let $N_t^n = \sum_{i=1}^n X_i 1_{\{\tau_i \leq t\}}$ and $N_t = \sum_{i=1}^{\infty} X_i 1_{\{\tau_i \leq t\}}$. The assumptions on N^n and N as of Theorem 6 are satisfied.

Lemma 9 *Under the assumptions above, $N_t \in L^1$ for each t , $N^n \xrightarrow{P} N$ and N has no fixed times of discontinuity.*

Proof. We prove the statement under assumption (i). For each t ,

$$E(|N_t|) \leq \sum_{i=1}^{\infty} E(|X_i| 1_{\{\tau_i \leq t\}}) \leq \mu \sum_{i=1}^{\infty} P(\tau_i \leq t) < \infty$$

Therefore, $N_t \in L^1$. For $\eta > 0$ and n integer, we obtain the following estimate.

$$\begin{aligned} P\left(\sup_{0 \leq t \leq T} |N_t - N_t^n| \geq \eta\right) &= P\left(\sup_{0 \leq t \leq T} \left| \sum_{i=n+1}^{\infty} X_i 1_{\{\tau_i \leq t\}} \right| \geq \eta\right) \\ &\leq P\left(\sup_{0 \leq t \leq T} \sum_{i=n+1}^{\infty} |X_i| 1_{\{\tau_i \leq t\}} \geq \eta\right) = P\left(\sum_{i=n+1}^{\infty} |X_i| 1_{\{\tau_i \leq T\}} \geq \eta\right) \\ &\leq \frac{1}{\eta} E\left(\sum_{i=n+1}^{\infty} |X_i| 1_{\{\tau_i \leq T\}}\right) = \frac{\mu}{\eta} \sum_{i=n+1}^{\infty} P(\tau_i \leq T) \rightarrow 0 \end{aligned}$$

This implies $N^n \xrightarrow{P} N$. Under assumption (ii), the proof is also straightforward and based on Cauchy Schwarz inequalities. Finally, since

$$P(|\Delta N_t| \neq 0) \leq P(\exists i \mid \tau_i = t) \leq \sum_{i=1}^{\infty} P(\tau_i = t) = 0$$

N has no fixed times of discontinuity ■

Since the random vector $(\tau_1, X_1, \dots, \tau_n, X_n)$ is assumed to satisfy Jacod's criterion, it follows from [13] that \mathbb{F} semimartingales remain \mathbb{G}^n semimartingales, for each n . Therefore, this property also holds between \mathbb{F} and \mathbb{G} for \mathbb{F} semimartingales whose \mathbb{G}^n canonical decompositions satisfy the regularity assumptions of Theorem 6. Here \mathbb{G} is the smallest filtration containing \mathbb{F} and to which N is adapted.

We would like to take a step further and reverse the previous situation. That is instead of starting with a sequence of processes N^n converging to some process N , and putting assumptions on the semimartingale properties of \mathbb{F} semimartingales w.r.t the intermediate filtrations \mathbb{G}^n and their decompositions therein, we would like to expand the filtration \mathbb{F} with a given process X and express all the assumptions in terms of X and the \mathbb{F} semimartingales considered. We are able to do this for càdlàg processes which satisfy a criterion that can loosely be seen as a localized extension of Jacod's criterion to processes. The integrability assumptions of Theorem 6 are expressed in terms of \mathcal{F}_t -conditional densities. Before doing this, we conclude this section by studying the stability of hypothesis (H) with respect to the weak convergence of the σ -fields in a filtration expansion setting.

3.3 The case of hypothesis (H)

Recall that given two nested filtrations $\mathbb{F} \subset \mathbb{G}$, we say that hypothesis (H) holds between \mathbb{F} and \mathbb{G} if any square integrable \mathbb{F} martingale remains an \mathbb{G} martingale. Brémaud and Yor proved the next Lemma 10 (see [4]).

Lemma 10 *Let $\mathbb{F} \subset \mathbb{G}$ two nested filtrations. The following assertions are equivalent.*

- (i) *Hypothesis (H) holds between \mathbb{F} and \mathbb{G} .*
- (ii) *For each $0 \leq t \leq T$, \mathcal{F}_T and \mathcal{G}_t are conditionally independent given \mathcal{F}_t .*
- (iii) *For each $0 \leq t \leq T$, each $F \in L^2(\mathcal{F}_T)$ and each $G_t \in L^2(\mathcal{G}_t)$,*

$$E(FG_t \mid \mathcal{F}_t) = E(F \mid \mathcal{F}_t)E(G_t \mid \mathcal{F}_t).$$

Let $\mathbb{F} \subset \mathbb{G}$ be two nested right-continuous filtrations and \mathbb{G}^n be a sequence of right-continuous filtrations containing \mathbb{F} and such that \mathcal{G}_t^n converges weakly to \mathcal{G}_t for each t . We mentioned that an \mathbb{F} local martingale that remains a \mathbb{G}^n semimartingale for each n might still lose its semimartingale property in \mathbb{G} and we provided conditions that prevent this pathological behavior. In this subsection, we prove that this cannot happen in case hypothesis (H) holds between \mathbb{F} and each \mathbb{G}^n . One obtains even that hypothesis (H) holds between \mathbb{F} and \mathbb{G} .

Theorem 7 *Let \mathbb{F} , \mathbb{G} and $(\mathbb{G}^n)_{n \geq 1}$ right-continuous filtrations such that $\mathbb{F} \subset \mathbb{G}$, $\mathbb{F} \subset \mathbb{G}^n$ for each n and $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$ for each t . Assume that for each n , hypothesis (H) holds between \mathbb{F} and \mathbb{G}^n . Then hypothesis (H) holds between \mathbb{F} and \mathbb{G} .*

Proof. We use Lemma 10 and start with the bounded case. Let $0 \leq t \leq T$, $F \in L^2(\mathcal{F}_T)$ and $G_t \in L^\infty(\mathcal{G}_t)$. For each n , define $G_t^n = E(G_t \mid \mathcal{G}_t^n)$. Then $G_t^n \in L^\infty(\mathcal{G}_t^n)$. Since hypothesis (H) holds between \mathbb{F} and \mathbb{G}^n , Lemma 10 guarantees that $E(FG_t^n \mid \mathcal{F}_t) = E(F \mid \mathcal{F}_t)E(G_t^n \mid \mathcal{F}_t)$. But $\mathcal{F}_t \subset \mathcal{G}_t^n$, hence $E(G_t^n \mid \mathcal{F}_t) = E(E(G_t \mid \mathcal{G}_t^n) \mid \mathcal{F}_t) = E(G_t \mid \mathcal{F}_t)$. Since $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$, $FG_t^n \xrightarrow{P} FG_t$. Now FG_t^n is bounded by a square integrable process (by assumption) so the convergence holds in L^1 by the Dominated Convergence theorem so that $E(FG_t^n \mid \mathcal{F}_t) \xrightarrow{P} E(FG_t \mid \mathcal{F}_t)$. This proves that $E(FG_t \mid \mathcal{F}_t) = E(F \mid \mathcal{F}_t)E(G_t \mid \mathcal{F}_t)$. The general case where $G_t \in L^2(\mathcal{G}_t)$ follows by applying the bounded case result to the bounded random variables $G_t^{(m)} = G_t \wedge m$. Then for each m ,

$$E(FG_t^{(m)} \mid \mathcal{F}_t) \xrightarrow{P} E(FG_t^{(m)} \mid \mathcal{F}_t)$$

and the Monotone Convergence theorem allows us to conclude. ■

4 Filtration expansion with a càdlàg process satisfying a *generalized Jacod's criterion* and applications to diffusions

In this section, we assume a càdlàg process X and a right-continuous filtration \mathbb{F} are given. We study the case where the process X and the filtration \mathbb{F} satisfy the following assumption.

Assumption 1 (Generalized Jacod's criterion) *There exists a sequence $(\pi_n)_{n \geq 1} = (\{t_i^n\})_{n \geq 1}$ of subdivisions of $[0, T]$ whose mesh tends to zero and such that for each n , $(X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \dots, X_T - X_{t_n^n})$ satisfies Jacod's criterion, i.e. there exists a σ -finite measure η_n on $\mathcal{B}(\mathbb{R}^{n+2})$ such that $P((X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \dots, X_T - X_{t_n^n}) \in \cdot \mid \mathcal{F}_t)(\omega) \ll \eta_n(\cdot)$ a.s.*

Under Assumption 1, the \mathcal{F}_t -conditional density

$$p_t^{(n)}(u_0, \dots, u_{n+1}, \omega) = \frac{P((X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \dots, X_T - X_{t_n^n}) \in (du_0, \dots, du_{n+1}) \mid \mathcal{F}_t)(\omega)}{\eta_n(du_0, \dots, du_{n+1})}$$

exists for each n , and can be chosen so that $(u_0, \dots, u_{n+1}, \omega, t) \rightarrow p_t^{(n)}(u_0, \dots, u_{n+1}, \omega)$ is càdlàg in t and measurable for the optional σ -field associated with the filtration $\hat{\mathbb{F}}_t$ given by $\hat{\mathcal{F}}_t = \cap_{u > t} \mathcal{B}(\mathbb{R}^{n+2}) \otimes \mathcal{F}_u$. For each $0 \leq i \leq n$, define

$$p_t^{i,n}(u_0, \dots, u_i) = \int_{\mathbb{R}^{n+1-i}} p_t^{(n)}(u_0, \dots, u_{n+1}) \eta_n(du_{i+1}, \dots, du_{n+1})$$

Let M be a continuous \mathbb{F} local martingale. Define

$$A_t^{i,n} = \int_0^t \frac{d\langle p^{i,n}(u_0, \dots, u_i), M \rangle_s}{p_s^{i,n}(u_0, \dots, u_i)} \Big|_{\forall 0 \leq k \leq i, u_k = X_{t_k^n} - X_{t_{k-1}^n}} \quad (5)$$

Finally define

$$A_t^{(n)} = \sum_{i=0}^n \int_{t \wedge t_i^n}^{t \wedge t_{i+1}^n} dA_s^{i,n}$$

i.e.

$$A_t^{(n)} = \sum_{i=0}^n 1_{\{t_i^n \leq t < t_{i+1}^n\}} \left(\sum_{k=0}^{i-1} \int_{t_k^n}^{t_{k+1}^n} dA_s^{k,n} + \int_{t_i^n}^t dA_s^{i,n} \right) \quad (6)$$

Of course, on each time interval $\{t_i^n \leq t < t_{i+1}^n\}$, only one term appears in the outer sum. Let \mathbb{G}^0 (resp. \mathbb{G}) be the smallest (resp. the smallest right-continuous) filtration containing \mathbb{F} and relative to which X is adapted. The theorem below is the main result of this section.

Theorem 8 *Assume X and \mathbb{F} satisfy Assumption 1 and that one of the following holds.*

- X has no fixed times of discontinuity,
- the sequence of subdivisions $(\pi_n)_{n \geq 1}$ in Assumption 1 is refining and each fixed time of discontinuity of X belongs to $\cup_n \pi_n$.

Let M be a continuous \mathbb{F} martingale such that $E(\sup_{s \leq T} |M_s|) \leq K$ and $E(\int_0^T |dA_s^{(n)}|) \leq K$ for some K and all n , with A^n as in (6). Then

- (i) M is a \mathbb{G}^0 special semimartingale.
- (ii) Moreover, if \mathbb{F} is the natural filtration of some càdlàg process Z , then M is a \mathbb{G} special semimartingale with canonical decomposition $M = N + A$ such that N is a \mathbb{G} martingale and $\sup_{0 \leq s \leq T} |N_s|$ and $\int_0^T |dA_s|$ are integrable.

Proof. We construct the discretized process X^n defined by $X_t^n = X_{t_k^n}$ for all $t_k^n \leq t < t_{k+1}^n$. That is

$$X_t^n = \sum_{i=0}^n X_{t_i^n} 1_{\{t_i^n \leq t < t_{i+1}^n\}} + X_T 1_{\{t=T\}}$$

with the convention $t_0^n = 0$ and $t_{n+1}^n = T$. Let \mathbb{G}^n be the smallest right-continuous filtration containing \mathbb{F} and to which X^n is adapted.

Now, for $0 \leq t \leq T$,

$$\begin{aligned} X_t^n &= \sum_{i=0}^n X_{t_i^n} 1_{\{t_i^n \leq t < t_{i+1}^n\}} + X_T 1_{\{t=T\}} = \sum_{i=0}^n X_{t_i^n} 1_{\{t_i^n \leq t\}} - \sum_{i=0}^n X_{t_{i+1}^n} 1_{\{t_{i+1}^n \leq t\}} + X_T 1_{\{t=T\}} \\ &= \sum_{i=1}^n (X_{t_i^n} - X_{t_{i-1}^n}) 1_{\{t_i^n \leq t\}} + X_0 1_{\{t_0^n \leq t\}} - X_{t_n^n} 1_{\{t_{n+1}^n \leq t\}} + X_T 1_{\{t_{n+1}^n \leq t\}} \\ &= X_0 1_{\{t_0^n \leq t\}} + \sum_{i=1}^{n+1} (X_{t_i^n} - X_{t_{i-1}^n}) 1_{\{t_i^n \leq t\}} = \sum_{i=0}^{n+1} (X_{t_i^n} - X_{t_{i-1}^n}) 1_{\{t_i^n \leq t\}} \end{aligned}$$

with the notation $X_{t_{-1}^n} = 0$.

For each $0 \leq i \leq n+1$, let $\mathbb{H}^{i,n}$ be the initial expansion of \mathbb{F} with $(X_{t_k^n} - X_{t_{k-1}^n})_{0 \leq k \leq i}$. Since $(X_{t_k^n} - X_{t_{k-1}^n})_{0 \leq k \leq i}$ satisfies Jacod's criterion, it follows that for each $0 \leq i \leq n+1$, $M - A^{i,n}$ is an $\mathbb{H}^{i,n}$ local martingale. Let

$$\tilde{\mathcal{G}}_t^n = \bigcap_{u>t} \mathcal{F}_u \vee \sigma((X_{t_i^n} - X_{t_{i-1}^n}) 1_{\{t_i^n \leq u\}}, i = 0, \dots, n+1)$$

Since the times t_k^n are fixed, $\mathbb{H}^{i,n}$ is also the initial expansion of \mathbb{F} with $(t_k^n, X_{t_k^n} - X_{t_{k-1}^n})_{0 \leq k \leq i}$ and $\tilde{\mathbb{G}}^n = \mathbb{G}^n$ using a Monotone Class argument and the fact that $X_{t_k^n}^n = X_{t_k^n}$, for all $0 \leq k \leq n+1$. So it follows from Theorem 8 in [13] that $M - A^{(n)}$ is a \mathbb{G}^n local martingale. An application of Theorem 6 yields the result. ■

We refrain from stating Theorem 8 in a more general form for clarity but provide two extensions in the remarks below.

- (i) Going beyond the continuous case for the \mathbb{F} local martingale M is straightforward. We only need to use Theorem 8 in [13] in its general version rather than its application to the continuous case. However the explicit form of $A^{(n)}$ is much more

complicated, which makes it hard to check the integrability assumption of Theorem 8. To be more concrete, one has to replace $A^{(n)}$ in the theorem above by $\tilde{A}^{(n)}$ defined by

$$\tilde{A}_t^{(n)} = \sum_{i=0}^n \int_{t \wedge t_i^n}^{t \wedge t_{i+1}^n} (d\tilde{A}_s^{i,n} + dJ_s^{i,n})$$

i.e.

$$\tilde{A}_t^{(n)} = \sum_{i=0}^n 1_{\{t_i^n \leq t < t_{i+1}^n\}} \left(\sum_{k=0}^{i-1} \int_{t_k^n}^{t_{k+1}^n} (d\tilde{A}_s^{k,n} + dJ_s^{k,n}) + \int_{t_i^n}^t (d\tilde{A}_s^{i,n} + dJ_s^{i,n}) \right)$$

where $\tilde{A}^{i,n}$ is the compensator of M in $\mathbb{H}^{i,n}$ as given by Jacod's theorem (see Theorems VI.10 and VI.11 in [15]) and $J^{i,n}$ is the dual predictable projection of $\Delta M_{t_{i+1}^n} 1_{[t_{i+1}^n, \infty[}$ onto $\mathbb{H}^{i,n}$.

- (ii) A careful study of the proof above shows that Assumption 1 is only used to ensure that there exists an $\mathbb{H}^{i,n}$ predictable process $A^{i,n}$ such that $M - A^{i,n}$ is an $\mathbb{H}^{i,n}$ local martingale. Therefore, Theorem 8 will hold whenever this weaker assumption is satisfied.

If the sequence of filtrations \mathbb{G}^n converges weakly to \mathbb{G} then $(M - A^{(n)}, A^{(n)})$ converges in probability under the Skorohod J_1 topology to (N, A) . Many criteria for this to hold are provided in the literature, see for instance Propositions 3 and 4 in [6]. This holds for example when every \mathbb{G} martingale is continuous and the subdivision $(\pi_n)_{n \geq 1}$ is refining. In this case, for each $0 \leq t \leq T$, $(\mathcal{G}_t^n)_{n \geq 1}$ is increasing and converges weakly to the σ -field \mathcal{G}_t . The following lemma allows us to conclude. See [6] for a proof.

Lemma 11 *Assume that every \mathbb{G} martingale is continuous and that for every $0 \leq t \leq T$, $(\mathcal{G}_t^n)_{n \geq 1}$ increases (or decreases) and converges weakly to \mathcal{G}_t . Then $\mathbb{G}^n \xrightarrow{w} \mathbb{G}$.*

4.1 Application to diffusions

Start with a Brownian filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, $\mathcal{F}_t = \sigma(B_s, s \leq t)$ and consider the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

Assume the existence of a unique strong solution $(X_t)_{0 \leq t \leq T}$. Indeed, assume the transition density $\pi(t, x, y)$ exists and is twice continuously differentiable in x and continuous in t and y . This is guaranteed for example if b and σ are infinitely differentiable with bounded derivatives and if the Hörmander condition holds for any x (see [2]), and we assume that this holds in the sequel. In this case, π is even infinitely differentiable.

We next show how we can expand a filtration dynamically as t increases, via another stochastic process evolving backwards in time. To this end, define the time reversed process $Z_t = X_{T-t}$, for all $0 \leq t \leq T$. Let $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t < \frac{T}{2}}$ be the smallest right-continuous filtration containing $(\mathcal{F}_t)_{0 \leq t < \frac{T}{2}}$ and to which $(Z_t)_{0 \leq t < \frac{T}{2}}$ is adapted. We would like to

prove that B remains a special semimartingale in \mathbb{G} and give its canonical decomposition. That B is a \mathbb{G} semimartingale can be obtained using the usual results from the filtration expansion theory. However, our approach allows us to obtain the decomposition, too. We assume (w.l.o.g) that $T = 1$. Introduce the reversed Brownian motion $\tilde{B}_t = B_{1-t} - B_1$ and the filtration $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_t)_{0 \leq t < \frac{1}{2}}$ defined by

$$\tilde{\mathcal{G}}_t = \bigcap_{t < u < \frac{1}{2}} \sigma(B_s, \tilde{B}_s, 0 \leq s < u).$$

Theorem 9 *Both B and \tilde{B} are \mathbb{G} semimartingales.*

Proof. First, it is well known that \tilde{B} is a Brownian motion in its own natural filtration and $\sigma(B_{1-s} - B_1, 0 \leq s < \frac{1}{2})$ is independent from $\sigma(B_s, 0 \leq s < \frac{1}{2})$. Therefore $(B_t)_{0 \leq t < \frac{1}{2}}$ and $(\tilde{B}_t)_{0 \leq t < \frac{1}{2}}$ are independent Brownian motions in $\tilde{\mathbb{G}}$. Now, given our strong assumptions on the coefficients b and σ , X_1 satisfies Jacod's criterion with respect to $\tilde{\mathbb{G}}$. Therefore B and \tilde{B} remain semimartingales in $\mathbb{H} = (\mathcal{H}_t)_{0 \leq t < \frac{1}{2}}$ where $\mathcal{H}_t = \bigcap_{\frac{1}{2} > u > t} \tilde{\mathcal{G}}_u \vee \sigma(X_1)$. It only remains to prove that $\mathbb{G} = \mathbb{H}$. For this, use Theorem V.23 in [15] to get that

$$dX_{1-t} = \sigma(X_{1-t})d\tilde{B}_t + (\sigma'(X_{1-t})\sigma(X_{1-t}) + b(X_{1-t}))dt$$

Since $b + \sigma\sigma'$ and σ are Lipschitz, $\bigcap_{\frac{1}{2} > u > t} \sigma(X_{1-s}, 0 \leq s \leq u) = \bigcap_{\frac{1}{2} > u > t} \sigma(\tilde{B}_s, 0 \leq s \leq u) \vee \sigma(X_1)$ and the result follows. ■

We apply now our results to obtain the \mathbb{G} decomposition. This is the primary result of this article.

Theorem 10 *Assume there exists a nonnegative function ϕ such that $\int_0^1 \phi(s)ds < \infty$ and for each $0 \leq s < t$,*

$$E\left(\left|\frac{1}{\pi} \frac{\partial \pi}{\partial x}(t-s, X_s, X_t)\right|\right) \leq \phi(t-s)$$

Then the process $(B_t)_{0 \leq t < \frac{1}{2}}$ is a \mathbb{G} semimartingale and

$$B_t - \int_0^t \frac{1}{\pi} \frac{\partial \pi}{\partial x}(1-2s, X_s, X_{1-s})ds$$

is a \mathbb{G} Brownian motion.

Proof. Since the process Z_t is a càdlàg process with no fixed times of discontinuity, we can apply Theorem 8. First we prove that $(Z_t)_{0 \leq t < \frac{1}{2}}$ and $(\mathcal{F}_t)_{0 \leq t < \frac{1}{2}}$ satisfy Assumption 1. Let $(\pi_n)_{n \geq 1} = (\{t_i^n\})_{n \geq 1}$ be a refining sequence of subdivisions of $[0, \frac{1}{2}]$ whose mesh tends to zero. We will do more and compute directly the conditional distributions of $(Z_{t_0^n}, Z_{t_1^n} - Z_{t_0^n}, \dots, Z_{t_i^n} - Z_{t_{i-1}^n})$ for any $1 \leq i \leq n+1$. Pick such i and let $0 \leq t < \frac{1}{2}$

and $(z_0, \dots, z_i) \in \mathbb{R}^{i+1}$.

$$\begin{aligned}
P(Z_{t_0^n} \leq z_0, Z_{t_1^n} - Z_{t_0^n} \leq z_1, \dots, Z_{t_i^n} - Z_{t_{i-1}^n} \leq z_i \mid \mathcal{F}_t) \\
&= P(X_1 \leq z_0, X_1 - X_{1-t_1^n} > -z_1, \dots, X_{1-t_{i-1}^n} - X_{1-t_i^n} > -z_i \mid \mathcal{F}_t) \\
&= E\left(\prod_{k=1}^{i-1} 1_{\{X_{1-t_k^n} - X_{1-t_{k+1}^n} \geq z_{k+1}\}} P(X_{1-t_1^n} - z_1 \leq X_1 \leq z_0 \mid \mathcal{F}_{1-t_1^n}) \mid \mathcal{F}_t\right) \\
&= E\left(\prod_{k=1}^{i-1} 1_{\{X_{1-t_k^n} - X_{1-t_{k+1}^n} \geq z_{k+1}\}} \int_{X_{1-t_1^n} - z_1}^{\infty} 1_{\{u_1 \leq z_0\}} P_{X_{1-t_1^n}}(t_1^n, u_1) du_1 \mid \mathcal{F}_t\right) \\
&= E\left(\prod_{k=1}^{i-1} 1_{\{X_{1-t_k^n} - X_{1-t_{k+1}^n} \geq z_{k+1}\}} \int_{-z_1}^{\infty} 1_{\{v_1 \leq z_0 - X_{1-t_1^n}\}} P_{X_{1-t_1^n}}(t_1^n, v_1 + X_{1-t_1^n}) dv_1 \mid \mathcal{F}_t\right)
\end{aligned}$$

Repeating the same technique and conditioning successively w.r.t $\mathcal{F}_{1-t_2^n}, \dots, \mathcal{F}_{1-t_i^n}$ gives

$$\begin{aligned}
P(Z_{t_0^n} \leq z_0, Z_{t_1^n} - Z_{t_0^n} \leq z_1, \dots, Z_{t_i^n} - Z_{t_{i-1}^n} \leq z_i \mid \mathcal{F}_t) &= E\left(\int_{-z_i}^{\infty} \dots \int_{-z_1}^{\infty} \right. \\
&\quad \left. 1_{\{\sum_{k=1}^i v_k \leq z_0 - X_{1-t_i^n}\}} \prod_{k=1}^i P_{X_{1-t_i^n} + \sum_{j=k+1}^i v_j}(t_k^n - t_{k-1}^n, \sum_{l=k}^i v_l + X_{1-t_i^n}) dv_1 \dots dv_i \mid \mathcal{F}_t\right) \\
&= \int_{-\infty}^{\infty} \int_{-z_i}^{\infty} \dots \int_{-z_1}^{\infty} 1_{\{u + \sum_{k=1}^i v_k \leq z_0\}} P_{X_t}(1 - t_i^n - t, u) \\
&\quad \prod_{k=1}^i P_{u + \sum_{j=k+1}^i v_j}(t_k^n - t_{k-1}^n, u + \sum_{l=k}^i v_l) dv_1 \dots dv_i du
\end{aligned}$$

Fubini's Theorem implies then

$$\begin{aligned}
P(Z_{t_0^n} \leq z_0, Z_{t_1^n} - Z_{t_0^n} \leq z_1, \dots, Z_{t_i^n} - Z_{t_{i-1}^n} \leq z_i \mid \mathcal{F}_t) &= \int_{-z_i}^{\infty} \dots \int_{-z_1}^{\infty} \int_{-\infty}^{z_0 - \sum_{k=1}^i v_k} \\
&\quad P_{X_t}(1 - t_i^n - t, u) \prod_{k=1}^i P_{u + \sum_{j=k+1}^i v_j}(t_k^n - t_{k-1}^n, u + \sum_{l=k}^i v_l) du dv_1 \dots dv_i
\end{aligned}$$

Since the transition density $\pi(t, x, y) = P_x(t, y)$ is twice continuously differentiable in x by assumption, it is straightforward to check that

$$p_t^{i,n}(z_0, \dots, z_i) = \prod_{k=1}^i \pi(t_k^n - t_{k-1}^n, \sum_{j=0}^k z_j, \sum_{j=0}^{k-1} z_j) \pi(1 - t_i^n - t, X_t, \sum_{j=0}^i z_j)$$

One then readily obtains

$$d\langle p_t^{i,n}(z_0, \dots, z_i), B. \rangle_s = \frac{1}{\pi} \frac{\partial \pi}{\partial x}(1 - t_i^n - s, X_s, \sum_{j=0}^i z_j) p_s^{i,n}(z_0, \dots, z_i) ds$$

Hence by taking the local martingale M in (5) to be B , we get

$$A_t^{i,n} = \int_0^t \frac{1}{\pi} \frac{\partial \pi}{\partial x} (1 - t_i^n - s, X_s, X_{1-t_i^n}) ds$$

Now equation (6) becomes

$$\begin{aligned} A_t^{(n)} = \sum_{i=0}^n 1_{\{t_i^n \leq t < t_{i+1}^n\}} & \left(\sum_{k=0}^{i-1} \int_{t_k^n}^{t_{k+1}^n} \frac{1}{\pi} \frac{\partial \pi}{\partial x} (1 - t_k^n - s, X_s, X_{1-t_k^n}) ds \right. \\ & \left. + \int_{t_i^n}^t \frac{1}{\pi} \frac{\partial \pi}{\partial x} (1 - t_i^n - s, X_s, X_{1-t_i^n}) ds \right) \end{aligned}$$

In order to apply Theorem 8, it only remains to prove that $E(\int_0^{\frac{1}{2}} |dA_s^{(n)}|) \leq K$ for some constant K independent from n . The finite constant $K = \int_0^1 \phi(s) ds$ works since

$$\begin{aligned} E\left(\int_0^{\frac{1}{2}} |dA_s^{(n)}|\right) & \leq \sum_{i=0}^n \int_{t_i^n}^{t_{i+1}^n} E\left|\frac{1}{\pi} \frac{\partial \pi}{\partial x} (1 - t_k^n - s, X_s, X_{1-t_k^n})\right| ds \\ & \leq \sum_{i=0}^n \int_{t_i^n}^{t_{i+1}^n} \phi(1 - t_k^n - s) ds = \sum_{i=0}^n \int_{1-t_k^n-t_{k+1}^n}^{1-2t_k^n} \phi(s) ds \\ & \leq \sum_{i=0}^n \int_{1-2t_{k+1}^n}^{1-2t_k^n} \phi(s) ds = \int_0^1 \phi(s) ds \end{aligned}$$

This proves again that B is a \mathbb{G} semimartingale. Now $A^{(n)}$ converges in probability to the process A given by

$$A_t = \int_0^t \frac{1}{\pi} \frac{\partial \pi}{\partial x} (1 - 2s, X_s, X_{1-s}) ds$$

Since all \mathbb{G} martingales are continuous, the comment following Theorem 8 ensures that $B-A$ is a \mathbb{G} martingale. Its quadratic variation is t , therefore it is a \mathbb{G} Brownian motion. \blacksquare

In the Brownian case, the result in Theorem 10 can also be obtained using the usual theory of initial expansion of filtration. Assume $b = 0$ and $\sigma = 1$, i.e. $Z = B_{1-}$ and $X = B$.

Theorem 11 *The process B is a \mathbb{G} semimartingale and*

$$B_t - \int_0^t \frac{B_{1-s} - B_s}{1 - 2s} ds, \quad 0 \leq t < \frac{1}{2}$$

is a \mathbb{G} Brownian motion.

Proof. Introduce the filtration $\mathbb{H}^1 = (\mathcal{H}_t)_{0 \leq t < \frac{1}{2}}$ obtained by initially expanding \mathbb{F} with $B_{\frac{1}{2}}$.

$$\mathcal{H}_t^1 = \bigcap_{u>t} F_u \vee \sigma(B_{\frac{1}{2}})$$

We know that B remains an \mathbb{H}^1 semimartingale and

$$M_t := B_t - \int_0^t \frac{B_{\frac{1}{2}} - B_s}{\frac{1}{2} - s} ds, \quad 0 \leq t < \frac{1}{2}$$

is an \mathbb{H}^1 Brownian motion. Now expand initially \mathbb{H}^1 with the independent σ -field $\sigma(B_v - B_{\frac{1}{2}}, \frac{1}{2} < v \leq 1)$ to obtain \mathbb{H} i.e.

$$\mathcal{H}_t = \bigcap_{u>t} H_u^1 \vee \sigma(B_v - B_{\frac{1}{2}}, \frac{1}{2} < v \leq 1)$$

Obviously $(M_t)_{0 \leq t < \frac{1}{2}}$ remains an \mathbb{H} Brownian motion. But $\mathcal{G}_t \subset \mathcal{H}_t$, for all $0 \leq t < \frac{1}{2}$, hence the optional projection of M onto \mathbb{G} , denoted oM in the sequel, is again a martingale (see [8]), i.e.

$${}^oM_t = B_t - E\left(\int_0^t \frac{B_{\frac{1}{2}} - B_s}{\frac{1}{2} - s} ds \mid \mathcal{G}_t\right), \quad 0 \leq t < \frac{1}{2}$$

is a \mathbb{G} martingale. Also, $N_t := E\left(\int_0^t \frac{B_{\frac{1}{2}} - B_s}{\frac{1}{2} - s} ds \mid \mathcal{G}_t\right) - \int_0^t E\left(\frac{B_{\frac{1}{2}} - B_s}{\frac{1}{2} - s} \mid \mathcal{G}_s\right) ds$ is a \mathbb{G} local martingale, see for example [13] for a proof. So

$$B_t = {}^oM_t + N_t + \int_0^t E\left(\frac{B_{\frac{1}{2}} - B_s}{\frac{1}{2} - s} \mid \mathcal{G}_s\right) ds$$

We prove now the theorem using properties of the Brownian bridge. Recall that for any $0 \leq T_0 < T_1 < \infty$,

$$\mathcal{L}\left((B_t)_{T_0 \leq t \leq T_1} \mid B_s, s \notin]T_0, T_1[\right) = \mathcal{L}\left((B_t)_{T_0 \leq t \leq T_1} \mid B_{T_0}, B_{T_1}\right) \quad (7)$$

and

$$\mathcal{L}\left((B_t)_{T_0 \leq t \leq T_1} \mid B_{T_0} = x, B_{T_1} = y\right) = \mathcal{L}\left(x + \frac{t - T_0}{T_1 - T_0}(y - x) + (Y_{t-T_0}^{W, T_1-T_0})_{T_0 \leq t \leq T_1}\right)$$

where W is a generic standard Brownian motion and Y^{W, T_1-T_0} is the standard Brownian bridge on $[0, T_1 - T_0]$. It follows that for all $T_0 \leq t \leq T_1$ and all x and y ,

$$E(B_t \mid B_{T_0} = x, B_{T_1} = y) = \frac{T_1 - t}{T_1 - T_0}x + \frac{t - T_0}{T_1 - T_0}y \quad (8)$$

For any $0 \leq s < t < \frac{1}{2}$, it follows from (7) and (8) that

$$E(B_{\frac{1}{2}} - B_s \mid \mathcal{G}_s) = \frac{1}{2}(B_{1-s} - B_s)$$

Therefore

$$B_t - \int_0^t \frac{B_{1-s} - B_s}{1 - 2s} ds = {}^oM_t + N_t$$

is a \mathbb{G} local martingale. Since the quadratic variation of the \mathbb{G} local martingale $B - A$ is t , Levy's characterization of Brownian motion ends the proof. ■

In the immediately previous proof, the properties of the Brownian bridge allow us to compute explicitly the decomposition of B in \mathbb{G} . Our method obtains both the semimartingale property and the decomposition simultaneously and generalizes to diffusions, for which the computations as in the proof of Theorem 11 are hard. We provide a shorter proof for Theorem 11 based on Theorem 10. This illustrated that, given Theorem 10, even in the Brownian case our method is shorter, simpler, and more intuitive.

Proof. [Second proof of Theorem 11] In the Brownian case, $\pi(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$. Therefore $\frac{1}{\pi} \frac{\partial \pi}{\partial x}(t, x, y) = \frac{y-x}{t}$. Hence

$$E\left(\left|\frac{1}{\pi} \frac{\partial \pi}{\partial x}\right|(t-s, B_s, B_t)\right) \leq \frac{1}{t-s} E(|B_t - B_s|) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{t-s}}$$

and $\phi(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}}$ is integrable in zero. From the closed formula for the transition density, $A_t = \int_0^t \frac{B_{1-s} - B_s}{1-2s} ds$. Therefore B is a \mathbb{G} semimartingale, and $B - A$ is a \mathbb{G} Brownian motion by Theorem 10. ■

This property satisfied by Brownian motion is inherited by diffusions whose parameters b and σ satisfy some boundedness assumptions. We add the extra assumptions that b and σ are bounded and $k \leq \sigma(x)$ for some $k > 0$. The following holds.

Corollary 3 *The process $(B_t)_{0 \leq t < \frac{1}{2}}$ is a \mathbb{G} semimartingale and*

$$B_t - \int_0^t \frac{1}{\pi} \frac{\partial \pi}{\partial x}(1-2s, X_s, X_{1-s}) ds$$

is a \mathbb{G} Brownian motion.

Proof. Introduce the following quantities

$$s(x) = \int_0^x \frac{1}{\sigma(y)} dy \quad g = s^{-1} \quad \mu = \frac{b}{\sigma} \circ g - \frac{1}{2} \sigma' \circ g$$

The process $Y_t = s(X_t)$ satisfies the SDE $dY_t = \mu(Y_t)dt + dB_t$. The transition density is known in semi-closed form (see [7]) and given by

$$\pi(t, x, y) = \frac{1}{\sqrt{2\pi t}} \frac{1}{\sigma(y)} e^{-\frac{(s(y)-s(x))^2}{2t}} U_t(s(x), s(y))$$

where $U_t(x, y) = H_t(x, y) e^{A(y) - A(x)}$, $H_t(x, y) = E(e^{-t \int_0^1 h(x+z(y-x) + \sqrt{t} W_z) dz})$, W is a Brownian bridge, A a primitive of μ and $h = \frac{1}{2}(\mu^2 + (\mu')^2)$. It is then straightforward to compute the ratio

$$\begin{aligned} \frac{1}{\pi} \frac{\partial \pi}{\partial x}(t, x, y) &= \frac{1}{\sigma(x)} \left(\frac{s(y) - s(x)}{t} + \frac{1}{U_t(s(x), s(y))} \frac{\partial U_t}{\partial x}(s(x), s(y)) \right) \\ &= \frac{1}{\sigma(x)} \left(\frac{s(y) - s(x)}{t} + \frac{1}{H_t(s(x), s(y))} \frac{\partial H_t}{\partial x}(s(x), s(y)) - \mu(s(x)) \right) \end{aligned}$$

From the boundedness assumptions of b and σ and their derivatives, there exists a constant M such that $|\frac{1}{\pi} \frac{\partial \pi}{\partial x}(t, x, y)| \leq M(1 + \frac{|s(y) - s(x)|}{t})$. Hence, for $0 \leq s < t$

$$E \left| \frac{1}{\pi} \frac{\partial \pi}{\partial x}(t - s, X_s, X_t) \right| \leq M \left(1 + E \left| \frac{s(X_t) - s(X_s)}{t - s} \right| \right)$$

But $s(X_t) - s(X_s) = Y_t - Y_s = \int_s^t \mu(Y_u) du + W_t - W_s$. But μ is bounded, hence

$$E|s(Y_t) - s(Y_s)| \leq \|\mu\|_\infty |t - s| + E|W_t - W_s| = \|\mu\|_\infty |t - s| + \sqrt{\frac{2}{\pi}} \sqrt{t - s}$$

This proves the existence of a constant C such that

$$E \left| \frac{1}{\pi} \frac{\partial \pi}{\partial x}(t - s, X_s, X_t) \right| \leq C \left(1 + \frac{1}{\sqrt{t - s}} \right)$$

Since $\phi(x) = C(1 + \frac{1}{\sqrt{x}})$ is integrable in zero, we can apply Theorem 10 and conclude. \blacksquare

4.2 Application to stochastic volatility models

Let $(\Omega, \mathcal{H}, P, \mathbb{H})$ be a filtered probability space. Assume that we are given an \mathbb{H} Brownian motion W and a positive continuous \mathbb{H} adapted process σ . Consider the following stochastic volatility model

$$dS_t = S_t \sigma_t dW_t$$

and σ is such that (σ, A) is Markov w.r.t its natural filtration with transition density $p_t((u, a), (v, b))$, where $A_t := \int_0^t \sigma_s^2 ds$. Define $Z_t = \int_0^t \sigma_s dW_s$, so that $S_t = \mathcal{E}(Z)_t$ and let \mathbb{F} be the filtration generated by S and σ . Then $\mathcal{F}_t = \bigcap_{u > t} \sigma(Z_s, \sigma_s, s \leq u)$. Since $\langle Z, Z \rangle_t = A_t$, A is \mathbb{F} adapted.

We want to expand $(\mathcal{F}_t)_{0 \leq t < \frac{T}{2}}$ progressively with the continuous process $X_t := \int_{T-t}^T \sigma_s^2 ds = A_T - A_t$, $0 \leq t < \frac{1}{2}$. The process $(X_t)_{0 \leq t < \frac{T}{2}}$ satisfies Assumption 1 with density

$$\int_{a_1=0}^{\infty} \int_{u_1=0}^{\infty} \cdots \int_{u_{n+1}=0}^{\infty} \prod_{k=1}^n p_{t_{k+1}^n - t_k^n}((u_k, A_t + \sum_{i=1}^k a_i), (u_{k+1}, A_t + \sum_{i=1}^{k+1} a_i)) \\ p_{\frac{T}{2}-t}((\sigma_t, A_t), u_1, a_1 + A_t)) du_{n+1} \dots du_1 da_1$$

along any refining subdivision $\{t_k^n\}$ of $[0, \frac{T}{2}]$. Let M be a continuous \mathbb{F} martingale such that $\sup_s |M_s|$ is integrable. Then $A^{i,n}$ and $A^{(n)}$ can be computed using the formulas in (5) and (6) and the \mathbb{G} semimartingale property of M will be guaranteed as soon as $E(\int_0^{\frac{T}{2}} |dA_s^{(n)}|) \leq K$ for some K and all n .

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